

# CONSTRUCTIONS OF GLOBAL INTEGRALS IN THE EXCEPTIONAL GROUPS

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**ABSTRACT.** Motivated by known examples of global integrals which represent automorphic  $L$ -functions, this paper initiates the study of a certain two-dimensional array of global integrals attached to any reductive algebraic group, indexed by maximal parabolic subgroups in one direction and by unipotent conjugacy classes in the other. Fourier coefficients attached to unipotent classes, Gelfand-Kirillov dimension of automorphic representations, and an identity which, empirically, appears to constrain the unfolding process are presented in detail with examples selected from the exceptional groups. Two new Eulerian integrals are included among these examples.

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## 1. Introduction

One of the important tools in the study of analytic properties of automorphic  $L$  functions is the method of integral representations. In this method one seeks to write down a global integral which depends on a complex variable  $s$ , show that this integral converges for  $\text{Re}(s)$  large, and has a continuation to the whole complex plane. Then the goal is to show that the integral is Eulerian, i.e. that it has an Euler product. In the literature, when the global integral contains an integration over a reductive group, this method is known as the Rankin-Selberg method. In the case when the integral is the Whittaker coefficient of an Eisenstein series, this method is known as the Langlands-Shahidi method. From the perspective which is adopted in this work, this division is not particularly natural, and hence we prefer to refer simply to “the method of integral representations.”

It is an easy task to write down an integral which has an analytic continuation. It is also easy to write global integrals which are Eulerian. However, to find integrals which satisfy both conditions is quite hard. We are still unaware of a general method of obtaining all integrals which satisfy both conditions.

In this paper we use ideas which were introduced in [G1] and [G3] in order to start a certain systematic approach for constructing examples of global integrals. We do it in the exceptional groups. There are two main reasons for that. First, the exceptional groups are not a part of an infinite family of groups. This enables us to check a finite list of cases. Second, the exceptional groups are less studied from these aspects than the classical groups. This gives the hope of possibly finding some interesting constructions. However, the methods introduced here are also applicable for the classical groups, and we do hope to study this issue in the future.

There are two main ingredients in our approach. The first is to use the classification of unipotent orbits (i.e., conjugacy classes consisting of unipotent elements) in order to write down global integrals. The main idea is as follows. Given a unipotent orbit  $\mathcal{O}$  in an

exceptional group  $H$ , we attach to it a parabolic subgroup  $P_{\mathcal{O}} = M_{\mathcal{O}}U_{\mathcal{O}}$  and a character  $\psi_{U_{\mathcal{O}}}$  defined on  $U_{\mathcal{O}}(F)\backslash U_{\mathcal{O}}(\mathbb{A})$  (or, in some cases, a subgroup). The stabilizer of  $\psi_{U_{\mathcal{O}}}$  in  $M_{\mathcal{O}}$  makes sense as an algebraic group defined over  $F$ , and it follows from the results in [C], that the identity component of this stabilizer is a reductive group  $C$ . Let  $E_{\tau}(h, s)$  denote an Eisenstein series defined on  $H(\mathbb{A})$ , associated with the induced representation  $\text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$ . See section 4 for details. Let  $\pi$  denote an irreducible cuspidal representation of  $C(\mathbb{A})$ . Then, under certain assumptions on the unipotent orbit  $\mathcal{O}$ , we can form the global integral

$$(1) \quad \int_{C(F)\backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}(F)\backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_{\pi}(g) E_{\tau}(ug, s) \psi_{U_{\mathcal{O}}}(u) du dg.$$

See sections 3 and 4 for details. Because of the cuspidality of  $\pi$ , this integral converges for  $\text{Re}(s)$  large and has at least a meromorphic continuation to the whole complex plane. Thus it clearly satisfies the first requirement we imposed above on our integrals. Notice that for certain unipotent orbits, the group  $C$  is trivial. In this case the above integral is just the Fourier coefficient of the Eisenstein series. When  $U_{\mathcal{O}}$  is the maximal unipotent subgroup of  $H$ , then  $\psi_{U_{\mathcal{O}}}$  is a generic character, and our construction specializes to the Langlands-Shahidi construction.

The next step is to address the problem of when an integral of the form given in equation (1) is Eulerian. This clearly depends on the representations  $\pi$  and  $\tau$ . Assume that  $\pi$  is a generic representation. A sufficient condition for the integral (1) to be Eulerian is that it is equal to an integral which involves the Whittaker function attached to  $\varphi_{\pi}$ , and some function in a model of  $\tau$  with similar factorization properties, integrated over a factorizable domain. We refer to such an integral as a “Whittaker integral.” Experience shows that if an integral of the type (1) unfolds to Whittaker integral, and is Eulerian, then a certain dimension equality holds. See [G3] for some more details. Therefore, we restrict to those integrals which satisfy this equation. This is all discussed in detail in section 4. We sketch the main ideas. To each of the representations  $\pi$  and  $E_{\tau}(\cdot, s)$  we attach a number which we refer to as the Gelfand-Kirillov dimension of the representation. The identity we require is

$$\dim C + \dim U_{\mathcal{O}} = \dim \pi + \dim E_{\tau}(\cdot, s)$$

This gives us an equation for  $\dim \tau$ , which then gives us certain information of which representation  $\tau$  to choose. To be clear, as far as we know, every global Eulerian integral of the type (1), which unfolds to a Whittaker coefficient of  $\pi$  satisfies the above dimension identity. However, there are global integrals, which satisfy this identity but do not unfold to a Whittaker coefficient of  $\pi$ . There are many such examples; we obtain some in section 7. Further, the question of whether an integral of the type (1) is Eulerian, and the question

of whether it unfolds to a Whittaker coefficient of  $\pi$  are two related, but separate questions. An answer to one does not necessarily determine an answer to the other either way.

The content of the paper is as follows. After fixing some notations we show how to attach a set of Fourier coefficients to a unipotent orbit of an exceptional group. This is done in section 3. In section 4 we construct the global integrals we intend to study and give a precise description of the basic dimension identity we use. There are several cases, and we discuss each of them. At the end of section 4 we write down a table of the condition which must be placed on the representation  $\tau$  in each case, in order for the dimension formula to be satisfied. We do this for the exceptional group  $F_4$ . This condition gives information about the Fourier coefficients supported by  $\tau$ . It is worthwhile to mention that in four of the rows in the table, the group  $C$  is trivial. In these cases, the integral (1) is just a Fourier coefficient of an Eisenstein series. The integrals obtained by considering the last row, are the Langlands-Shahidi integrals for the exceptional group  $F_4$ .

Section 5 is devoted to the general process of unfolding a global integral. We carry out this process in detail and as generally as we can. In this way we also fix some notation which will be of help to us later on. According to the general outcome, we partition the set of integrals into various types. Maybe the most interesting integrals are the ones we refer to as open orbit type integrals. See definition 29. These are those integrals, such that after some unfolding, we obtain as inner integration, an integral of a similar type to the one we started with, and which satisfies a similar dimension identity. This shows that in some sense the process is inductive. The study of the global integral is reduced to the study of a global integral defined on smaller groups.

Starting from section 6 we concentrate on two examples in the group  $F_4$ . These examples are typical and represent the general process. Therefore we study them in detail. In the notations of the table at the end of section 4, we consider in section 8 the global integral attached to the unipotent orbit  $\mathcal{O} = \tilde{A}_2$  and in section 10 the integral attached to the unipotent integral  $\mathcal{O} = A_1$ .

As mentioned above, this paper is a starting point towards a certain systematic study of constructing global integrals, first in the exceptional groups, and later in classical groups. Clearly there are many details which need to be addressed. First, in this project we assume that  $\pi$  is generic. However, one can study also the cases where the representation  $\pi$  is cuspidal and not generic. Secondly, the integrals which we construct here are natural in a certain sense. But one can also study global integrals, which satisfy the dimension identity given in section 4, but not of the type which we construct in this paper. Finally, there is the interesting question of determining which  $L$  functions we obtain for those integrals which

are Eulerian. This is, of course, a local question, and needs to be addressed by means of local computations. In the two examples we consider in this paper, we obtain two examples of such integrals which are new. We hope to study these topics in the future.

## 2. Basic notations

Let  $H$  denote one of the exceptional groups  $F_4$  or  $E_i$  for  $i = 6, 7, 8$ . By this we mean the unique split connected simply connected simple algebraic group of the given type defined over our number field  $F$ . We shall label the simple roots of  $H$  as follows. First, for the group  $H = F_4$ , the Dynkin is given by

$$\begin{array}{c} \alpha_1 \qquad \qquad \alpha_2 \qquad \qquad \alpha_3 \qquad \qquad \alpha_4 \\ 0 \text{ --- } 0 \text{ ==> } 0 \text{ --- } 0 \end{array}$$

As for the group  $H = E_8$ , we label the simple roots as

$$\begin{array}{cccccccccccc} \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 & & \alpha_7 & & \alpha_8 \\ 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 \\ & & & & | & & & & & & & & \\ & & & & 0 & & & & & & & & \\ & & & & \alpha_2 & & & & & & & & \end{array}$$

The Dynkin diagram for the groups  $H = E_6$  and  $H = E_7$ , are the ones derived from the diagram for  $E_8$  omitting the relevant roots.

For construction of several of our integrals, we will need to use the similitude groups for  $E_6$  and  $E_7$ . We shall denote these groups by  $GE_6$  and  $GE_7$ , and use the notations defined in [G2].

We assume  $H$  to be equipped with a choice of maximal torus  $T$  and Borel subgroup,  $B = TU_{\max}^H$ . Here  $U_{\max}^H$  is the unipotent radical of  $B$ , which is a maximal unipotent subgroup of  $H$ . If  $G$  is any  $T$ -stable subgroup of  $H$  we denote the set of roots of  $T$  in  $G$  by  $\Phi(G, T)$ . We also assume  $H$  to be equipped with a realization, in the sense of [Spr], i.e., a choice of isomorphism  $x_\alpha : \mathbb{G}_a \rightarrow U_\alpha$  for each root  $\alpha \in \Phi(H, T)$ , subject to certain compatibility conditions. Here  $U_\alpha$  is the one-dimensional subgroup of  $H$  attached to  $\alpha$ . We further assume that the structure constants for this realization are given as in [G-S].

For each root  $\alpha$ , the groups  $U_\alpha$  and  $U_{-\alpha}$  generate a subgroup of  $H$  which is isomorphic to  $SL_2$ . We denote this subgroup by  $SL_2^\alpha$ . Likewise, given two roots  $\alpha$  and  $\beta$ , the subgroup of  $H$  generated by  $U_{\pm\alpha}$  and  $U_{\pm\beta}$ , is denoted by  $SL_3^{\{\alpha, \beta\}}$  if it is isomorphic to  $SL_3$ , by  $Sp_4^{\{\alpha, \beta\}}$  if it is isomorphic to  $Sp_4$ , etc.

We shall denote the maximal parabolic subgroups of  $H$  as follows. Let  $P$  be a standard maximal parabolic subgroup of  $H$ . Let  $\alpha_i$  denote the unique simple root of  $H$  such that the one parameter unipotent subgroup  $x_{\alpha_i}(r)$  is in the unipotent radical of  $P$ . We shall then denote  $P$  by  $P_i$ . We also denote the unipotent radical by  $U_i$  and the standard Levi subgroup by  $M_i$ .

The choice of realization allows us to work with elements of  $H$  via explicit generators and relations. In this approach we shall refer to specific roots of  $H$  very often. If  $\alpha_1, \dots, \alpha_r$  are the simple roots of  $H$ , we identify the root  $\sum_{i=1}^r n_i \alpha_i$  with the tuple  $(n_1, \dots, n_r)$ . Since none of the coefficients  $n_i$  ever exceeds 9, we shall normally simply run the digits together. For example, the root  $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$  of  $F_4$  will be denoted 2342 or (2342).

*2.0.1. Some conventions for defining characters.* Suppose that  $U$  is a unipotent group and we need to define a character of  $U$ . This will occur very frequently in this work. We introduce two convenient notational conventions for this, at least in the case when  $U$  is  $T$ -stable, in which case it is a product of groups  $U_\alpha$ . Assume that this is so. Fix a total ordering on  $\Phi(U, T)$ . Then we may define an isomorphism of varieties  $\mathbb{G}_a^{\dim U} \rightarrow U$  by

$$(u_\alpha)_{\alpha \in \Phi(U, T)} \mapsto \prod_{\alpha \in \Phi(U, T)} x_\alpha(u_\alpha),$$

with the product defined using our fixed total ordering. This in effect defines “coordinates”  $(u_\alpha)_{\alpha \in \Phi(U, T)}$  on  $U$ . Further, the  $\alpha$ -coordinate  $u_\alpha$  of  $u \in U$  depends on the choice of total ordering only if  $U_\alpha$  lies in the commutator subgroup of  $U$ . In particular, characters may be written using these coordinates without ambiguity, even when the total ordering has not been specified. For example, one can define a character of the maximal unipotent subgroup of  $F_4$  by  $\psi_{U_{\max}}(u) = \psi(u_{0100} + u_{0010} + u_{0001})$ . Note that  $u_{1000}, u_{0100}, u_{0010}$  and  $u_{0001}$  are the only coordinates of  $u \in U_{\max}$  which are well defined, independently of the total ordering chosen. On the other hand, they are also the only coordinates on which a character of  $U_{\max}$  may depend. Finally, note that while these coordinates are independent of the choice of total order, they are very much dependent on the choice of realization  $\{x_\alpha : \alpha \in \Phi(H, T)\}$ . But the realization has been fixed once and for all, so this is no cause for concern.

Sometimes a different convention is more convenient. If we write

$$\psi_U(x_{\beta_1}(r_1)x_{\beta_2}(r_2)u') = \psi(r_1 + r_2),$$

or the like, the convention is that the restriction of  $U$  to  $U_\alpha$  is trivial for all  $\alpha \in \Phi(U, T)$  except for those listed (here  $\beta_1$ , and  $\beta_2$ ). Also  $u'$  is an element of the product of the groups  $U_\alpha, \alpha \in \Phi(U, T)$  with the listed roots excluded.

### 3. Construction of Fourier Coefficients

In this section we shall explain how to construct a Fourier coefficient from a weighted diagram corresponding to a unipotent orbit. Our construction is essentially a slight refinement of the one given in [G-R-S1], and applies to any split connected reductive algebraic group.

Our main interest are constructions in the exceptional groups, however this setup holds for classical groups as well. Let  $H$  denote a split connected reductive algebraic group.

We shall work in characteristic zero. Consequently, we may pass freely back and forth between unipotent orbits in a reductive group and nilpotent orbits in its Lie algebra. Indeed, the exponential map gives an equivariant isomorphism between the nilpotent and unipotent subvarieties. We may also pass back and forth between adjoint orbits in the Lie algebra and coadjoint orbits in its dual, as described in [C-M], section 1.3.

We use the classification results described in [C], particularly the tables of orbits which begin on page 401, as well as the Bala-Carter labels used therein. For the list of dimensions of these orbits, we use [C-M] page 128. A well known result of Dynkin which is described in section 5.6 of [C] associates with any unipotent orbit a Dynkin diagram whose roots are labeled with zeros, ones and twos, which determines the orbit completely. For exceptional groups, the Bala-Carter label or the weighted Dynkin diagram is the most common method of specifying an orbit. For classical groups it is more common to use partitions as in [G-R-S1]. However, the parametrization via weighted Dynkin diagrams is valid in general and, indeed, the first part of the construction given in [G-R-S1] amounts to recovering the weighted Dynkin diagram attached to an orbit from the partition.

Let  $\mathcal{O}$  denote a unipotent orbit for the group  $H$ . (We shall identify each algebraic group which we consider with its  $\overline{F}$  points where  $\overline{F}$  is a fixed algebraic closure of  $F$ .) First, we shall associate to  $\mathcal{O}$  a parabolic subgroup of  $H$  which we shall denote by  $P_{\mathcal{O}}$ . We shall write its Levi decomposition  $P_{\mathcal{O}} = M_{\mathcal{O}}U_{\mathcal{O}}$ . Here  $M_{\mathcal{O}}$  is the standard Levi subgroup of  $P_{\mathcal{O}}$  and  $U_{\mathcal{O}}$  is its unipotent radical.

The parabolic subgroup  $P_{\mathcal{O}}$  is defined to be the standard parabolic subgroup of  $H$  whose Levi part is generated by the maximal torus of  $H$ , and all copies of  $SL_2$  corresponding to those simple roots which are labeled by zero in the diagram. For example, in the group  $H = E_6$  the diagram corresponding to the unipotent class  $\mathcal{O} = 2A_2$  is given by

$$(2) \quad \begin{array}{ccccccc} \overset{2}{0} & - & 0 & - & 0 & - & 0 & - & \overset{2}{0} \\ & & & & | & & & & \\ & & & & 0 & & & & \end{array}$$

Here and throughout, roots without a number are considered as labeled with the number zero. In other words, the roots  $\alpha_2, \alpha_3, \alpha_4$  and  $\alpha_5$  are labeled with the number zero. Thus, in this case  $M_{\mathcal{O}} = \text{Spin}_8 \cdot GL_1^2$ .

As a second example, consider in  $H = F_4$  the unipotent class  $\mathcal{O} = B_2$ . Its diagram is given by

$$\overset{2}{0} - - - - 0 == > == 0 - - - - \overset{1}{0}$$

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Therefore, the Levi part of  $P_{\mathcal{O}}$  is given by  $M_{\mathcal{O}} = Sp_4 \cdot GL_1^2$ .

Since  $P_{\mathcal{O}}$  is standard,  $U_{\mathcal{O}}$  is generated by one dimensional unipotent subgroups  $x_{\alpha}(r)$ , where  $\alpha$  is positive. For example, in the case of diagram (2), the unipotent group  $U_{\mathcal{O}}$  is generated by all  $x_{\alpha}(r)$  such that if we write  $\alpha = \sum_{i=1}^6 n_i \alpha_i$ , then either  $n_1 > 0$  or  $n_6 > 0$ . Thus  $\dim U_{\mathcal{O}} = 48$ .

Let  $\psi$  denote a nontrivial character of  $F \backslash \mathbb{A}$ . To the unipotent orbit  $\mathcal{O}$  we shall associate a character  $\psi_{U_{\mathcal{O}}}$  defined on a subgroup of  $U_{\mathcal{O}}$ . In the case of the classical group this procedure is explained in detail in [G1] section two. For the exceptional groups this is done in a similar way. To describe the construction, we fix some notations.

We will say that the root  $\alpha$  is in  $U_{\mathcal{O}}$  if the one parameter unipotent subgroup  $x_{\alpha}(r)$  is in  $U_{\mathcal{O}}$ . Assume that the simple roots  $\alpha_{i_1}, \dots, \alpha_{i_r}$  are the simple roots which are labeled zero in the diagram corresponding to the unipotent orbit  $\mathcal{O}$ . By definition, a positive root  $\alpha = \sum_i n_i \alpha_i$  is in  $U_{\mathcal{O}}$  if and only if  $n_{i_1} + \dots + n_{i_r} > 0$ . For each  $k > 0$  we set  $U_{\mathcal{O}}^{(k)}$  equal to the product of the groups  $U_{\alpha}$  for  $\alpha$  in the set  $\{\alpha = \sum_i n_i \alpha_i \quad : \quad n_{i_1} + \dots + n_{i_r} \geq k\}$ . It is easy to see that  $U_{\mathcal{O}}^{(k)}$  is a group and that  $L_{\mathcal{O}}^{(k)} = U_{\mathcal{O}}^{(k)} / U_{\mathcal{O}}^{(k+1)}$  is an abelian group. We also define  $V_{\mathcal{O}}^{(k)}$  as the product of the groups  $U_{\alpha}$  corresponding to those roots  $\alpha = \sum_i n_i \alpha_i$  with  $\sum_i c_i n_i \geq k$ . Here,  $c_i \in \{0, 1, 2\}$  is the weight attached to the simple root  $\alpha_i$  in the weighted Dynkin diagram of  $\mathcal{O}$ . We consider the group  $V_{\mathcal{O}}^{(2)} / V_{\mathcal{O}}^{(3)}$ . It may equal  $L_{\mathcal{O}}^{(1)}$ , or  $L_{\mathcal{O}}^{(2)}$ , or some combination of the two. The group  $M_{\mathcal{O}}$  acts on this group with an open orbit. We say that an element of  $L_{\mathcal{O}}^{(2)}$  is “in general position” if it is in this orbit. See proposition 5.7.3 of [C] for details. It follows from proposition 5.5.9, p. 156 of [C] that the identity component of the stabilizer inside  $M_{\mathcal{O}}$  of a representative of the open orbit is a reductive group.

To define the character  $\psi_{U_{\mathcal{O}}}$ , assume first that the diagram attached to  $\mathcal{O}$  has only zeros and twos. In this case,  $V_{\mathcal{O}}^{(2)} / V_{\mathcal{O}}^{(3)} = L_{\mathcal{O}}^{(1)}$ , and  $\psi_{U_{\mathcal{O}}}$  will be a character of  $U_{\mathcal{O}}$  itself. We consider the action of  $M_{\mathcal{O}}$  on the group  $L_{\mathcal{O}}^{(1)}$ . This action is essentially a rational representation of  $M_{\mathcal{O}}$  and the  $F$ -points of the dual rational representation are identified with the set of all characters of  $U_{\mathcal{O}}$  by our choice of  $\psi$ . As explained above, over an algebraically closed field, the action of  $M_{\mathcal{O}}$  on  $L_{\mathcal{O}}^{(1)}$  has an open orbit, and the identity component of the stabilizer of any element of this orbit is reductive. The same is true of the action of  $M_{\mathcal{O}}$  on the dual rational representation, which we denote  $L_{\mathcal{O}}^{(1),*}$ . However, the action of  $M_{\mathcal{O}}(F)$  on  $L_{\mathcal{O}}^{(1),*}(F)$  need not have an open orbit. Indeed, the intersection of  $L_{\mathcal{O}}^{(1),*}(F)$  with the open orbit in  $L_{\mathcal{O}}^{(1)}$  may be a union of several  $M_{\mathcal{O}}(F)$  orbits, with stabilizers which are isomorphic to one another only over the closure and not over  $F$ . This issue is discussed in [G1] in example 2 page 328. For  $\psi_{U_{\mathcal{O}}}$  we choose a character which is in the open orbit. Following the tables given in [C] page 401, we shall denote the connected component of the stabilizer by  $C$ .



$$(3) \quad \varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}(g) := \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi(ug) \psi_{U_{\mathcal{O}}}(u) du \quad (g \in C(\mathbb{A}))$$

**Lemma 4.** The function  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  is smooth, and  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  and all of its derivatives have moderate growth.

We are also given that  $\varphi$  is of moderate growth. This is defined using a choice of embedding  $H \hookrightarrow GL_N$  for some  $N$  (though the condition obtained is independent of the choice of embedding). If we then define “moderate growth for  $C$  using the embedding  $C \hookrightarrow H \hookrightarrow GL_N$ , then it is clear that  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  inherits this property as well.  $\square$

We consider a few examples. First, let  $H = E_6$  and  $\mathcal{O} = 2A_2$ . The roots in  $L_{\mathcal{O}}^{(1)}$  are all those roots  $\alpha = \sum_{i=1}^6 n_i \alpha_i$  such that  $n_1 + n_6 = 1$ . There are 16 such roots. The group  $M_{\mathcal{O}} = \text{Spin}_8 \cdot GL_1^2$  acts on this group. The representation is reducible, and up to the action of  $GL_1^2$  it is equal to two of the three eight dimensional representations of  $\text{Spin}_8$ . It follows from [C] that the stabilizer of the open orbit is the exceptional group  $G_2$ . For  $u \in U_{\mathcal{O}}$  define

where  $u' \in U_{\mathcal{O}}$  is any product of unipotent elements  $x_{\alpha}(r)$  in  $U_{\mathcal{O}}$  such that  $\alpha$  is not one of the above four roots. Then the stabilizer inside  $M_{\mathcal{O}}$  is the exceptional group  $G_2$ .

$$\begin{array}{ccccccccccc}
 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0^2 & \text{---} & 0^2 \\
 & & & & | & & & & & & & & \\
 & & & & 0 & & & & & & & & 
 \end{array}$$

In this case  $L_{\mathcal{O}}^{(1)}$  consists of all roots  $\alpha = \sum_{i=1}^8 n_i \alpha_i$  such that  $n_7 + n_8 = 1$ . There are 28 such roots. The action of  $M_{\mathcal{O}} = E_6 \cdot GL_1^2$  on the roots in  $L_{\mathcal{O}}^{(1)}$  gives a sum of two irreducible representations. First, on  $x_{\alpha_8}(r)$  we obtain a one dimensional representation of  $M_{\mathcal{O}}$ . The second representation corresponds to the 27 roots  $\alpha = \sum_{i=1}^8 n_i \alpha_i$  such that  $n_7 = 1$  and  $n_8 = 0$ , is the 27 dimensional representation of  $E_6$ . If we define the character

$$\psi_{U_{\mathcal{O}}}(u) = \psi_{U_{\mathcal{O}}}(x_{00000001}(r_1)x_{11221110}(r_2)x_{11122210}(r_3)x_{01122210}(r_4)u') = \psi(r_1 + r_2 + r_3 + r_4)$$

then the stabilizer in  $M_{\mathcal{O}}$  is the exceptional group  $F_4$ .

As a final example of this type, suppose that the diagram attached to  $\mathcal{O}$  contains only twos and no zeros. Then  $M_{\mathcal{O}}$  is the torus of the group  $H$  and  $U_{\mathcal{O}}$  is the maximal unipotent subgroup of  $H$ . Therefore, in this case we may take  $\psi_{U_{\mathcal{O}}}$  the standard generic character corresponding to our chosen realization. In other words, if  $\alpha_i$  are the simple roots of  $H$ , then

$$\psi_{U_{\mathcal{O}}}(u) = \psi_{U_{\mathcal{O}}}(x_{\alpha_1}(r_1) \dots x_{\alpha_m}(r_m)u') = \psi(r_1 + \dots + r_m).$$

Next, we consider the case when the diagram attached to the unipotent orbit contains only zeros and ones. In this case we define the character on the unipotent subgroup  $U_{\mathcal{O}}^{(2)}$ . More precisely, we consider the action of  $M_{\mathcal{O}}$  on the group  $L_{\mathcal{O}}^{(2)}$ , and we define the character  $\psi_{U_{\mathcal{O}}^{(2)}}$  in a similar way as we defined the character  $\psi_{U_{\mathcal{O}}}$  in the previous case.

As a first example we consider in the group  $H = E_6$  the unipotent orbit  $\mathcal{O} = A_1$ . The diagram which corresponds to this unipotent class is

$$(5) \quad \begin{array}{ccccccc} 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 & \text{---} & 0 \\ & & & & | & & & & \\ & & & & 0 & & & & \\ & & & & 1 & & & & \end{array}$$

In this case, the group  $U_{\mathcal{O}}$  is a Heisenberg group, and the group  $L_{\mathcal{O}}^{(2)}$  has only one root, which is (122321). Therefore, the character we attach to this unipotent class is given by  $\psi_{U_{\mathcal{O}}^{(2)}}(x_{122321}(r)) = \psi(r)$ . In this case, the connected component of the stabilizer  $C$  is a group of type  $A_5$ . As in integral (3), given an automorphic representation  $\sigma$  of  $E_6(\mathbb{A})$ , we can define the Fourier coefficient which corresponds to this orbit by

$$(6) \quad \int_{U_{\mathcal{O}}^{(2)}(F) \backslash U_{\mathcal{O}}^{(2)}(\mathbb{A})} \varphi_{\sigma}(ug) \psi_{U_{\mathcal{O}}^{(2)}}(u) du.$$

This Fourier coefficient defines an automorphic function defined on the group  $C(\mathbb{A})$ .

As an another example, consider the unipotent class  $\mathcal{O} = \tilde{A}_1$  in  $H = F_4$ . Its diagram is

$$(7) \quad \begin{array}{ccccccc} 0 & \text{---} & \text{---} & \text{---} & -0 & \text{==>==} & 0 & \text{---} & \text{---} & \text{---} & -0 \\ & & & & & & & & & & 1 \\ & & & & & & & & & & 10 \end{array}$$

In this case the group  $U_{\mathcal{O}}^{(2)}$  consists of the seven roots  $\alpha = \sum_{i=1}^4 n_i \alpha_i$  in  $H$  such  $n_4 = 2$ . If we define the character

$$\psi_{U_{\mathcal{O}}^{(2)}}(u) = \psi_{U_{\mathcal{O}}^{(2)}}(x_{1232}(r)u') = \psi(r),$$

then the connected component of the stabilizer inside  $M_{\mathcal{O}}$  is the group  $\text{Spin}_6$ .

Returning to the general case of diagrams which contains zeros and ones only, it will be convenient to extend the unipotent group we integrate over. This extension, which involves the theta representation defined on the double cover of a suitable symplectic group, is discussed in [G-R-S1] section 1 in detail for unipotent orbits of the symplectic group. In the exceptional groups it is exactly the same. The group  $U_{\mathcal{O}}/U_{\mathcal{O}}^{(3)}$  has the structure of a generalized Heisenberg group. (By this we mean a group which has a projection onto a Heisenberg group such that the kernel is contained in the center. See section 8 for a more detailed definition.) This is a general phenomenon, which we explain briefly in section 8. For example, in the case of the unipotent class  $A_1$  in  $E_6$ , the group  $U_{\mathcal{O}}^{(3)}$  is trivial, and  $U_{\mathcal{O}}$  is the Heisenberg group with 21 variables. The action of the group  $C$  on  $U_{\mathcal{O}}$  by conjugation therefore induces a homomorphism into certain symplectic group. In the above example, it is the group  $Sp_{20}$ .

In general there is a projection map  $l : U_{\mathcal{O}}/U_{\mathcal{O}}^{(3)} \rightarrow \mathcal{H}_m$  where  $\mathcal{H}_m$  is a Heisenberg group with  $m$  variables. Here  $m = \dim L_{\mathcal{O}}^{(1)} + 1$ . As an example to this phenomena, consider in  $H = F_4$ , the unipotent class  $\mathcal{O} = A_1 + \tilde{A}_1$ . Its diagram is given by

$$0 - - - \overset{1}{-0} ==> == 0 - - - -0$$

In this case, the roots in  $L_{\mathcal{O}}^{(1)}$  are all twelve roots  $\alpha = \sum_{i=1}^4 n_i \alpha_i$  such that  $n_2 = 1$ , and the roots in  $L_{\mathcal{O}}^{(2)}$  are the six roots with  $n_2 = 2$ . The group  $U_{\mathcal{O}}/U_{\mathcal{O}}^{(3)}$  is a generalized Heisenberg group, and in this case we define  $l : U_{\mathcal{O}}/U_{\mathcal{O}}^{(3)} \rightarrow \mathcal{H}_{13}$  as follows. Recall that  $\mathcal{H}_{13}$  is  $\mathbb{G}_a^6 \times \mathbb{G}_a^6 \times \mathbb{G}_a$  equipped with the operation

$$(x_1|y_1|z_1) \cdot (x_2|y_2|z_2) = \left( x_1 + x_2 \mid y_1 + y_2 \mid z_1 + z_2 + \frac{1}{2}(x_1 \cdot {}_t y_2 - y_1 \cdot {}_t x_2) \right).$$

Here  $x_1, y, x_2, y_2 \in \mathbb{G}_a^6$ , realized as row vectors, and  ${}_t$  denotes the lower transpose:

$${}_t(x_1 \dots x_n) = \begin{pmatrix} x_n \\ \vdots \\ x_1 \end{pmatrix}.$$

Also  $z_1, z_2 \in \mathbb{G}_a$ . (We separate the components of  $\mathcal{H}_{13}$  with vertical bars to aid legibility in computations where individual entries in the row vectors are written out.)

The mapping of  $L_{\mathcal{O}}^{(2)}$  onto the center of  $\mathcal{H}_{13}$  should be an element of  $L_{\mathcal{O}}^{(2),*}$  (the  $M_{\mathcal{O}}$ -module dual to  $L_{\mathcal{O}}^{(2)}$ ) in general position. One option is as follows. For any  $u' \in U_{\mathcal{O}}^{(3)}$  we have

$$l(x_{1220}(r_1)x_{1221}(r_2)x_{1222}(r_3)x_{1231}(r_4)x_{1232}(r_5)x_{1242}(r_6)u') = (0|0|r_3 + r_4)$$

It is not hard to check that the stabilizer inside  $M_{\mathcal{O}}$  is a group of type  $A_1 + A_1$  as indicated in the table in [C].

In conjunction with the commutator map  $L_{\mathcal{O}}^{(1)} \times L_{\mathcal{O}}^{(1)} \rightarrow L_{\mathcal{O}}^{(2)}$ , the linear form on  $L_{\mathcal{O}}^{(2)}$  which has been chosen determines a skew-symmetric form on  $L_{\mathcal{O}}^{(1)} \times L_{\mathcal{O}}^{(1)}$ . To extend it to a projection  $U_{\mathcal{O}} \rightarrow \mathcal{H}_{13}$ , one needs an isomorphism  $L_{\mathcal{O}}^{(1)} \rightarrow \mathbb{G}_a^6 \times \mathbb{G}_a^6$  such that the preimage of  $\{(x|0|0) : x \in \mathbb{G}_a^6\}$  in  $L_{\mathcal{O}}^{(1)}$  is isotropic with respect to this skew-symmetric form, and so is the preimage of  $\{(0|y|0) : y \in \mathbb{G}_a^6\}$ .

In the case at hand, one might take

$$l(x_{0100}(m_1)x_{1100}(m_2)x_{0110}(m_3)x_{1110}(m_4)x_{0111}(m_5)x_{0120}(m_6)) = (X|0|0)$$

where  $X = (m_1, m_2, \dots, m_6) \in \mathbb{G}_a^6$ . Then  $l$  will map

$$\{x_{1120}(m_1)x_{1111}(m_2)x_{0121}(m_3)x_{1121}(m_4)x_{0122}(m_5)x_{1122}(m_6) : m_1, \dots, m_6\}$$

isomorphically onto  $\{(0|Y|0) : Y \in \mathbb{G}_a^6\} \subset \mathcal{H}_{13}$ . The precise isomorphism will depend on the structure constants of  $F_4$ . This issue is discussed in more detail later on in section 8.

Returning to the general case, having fixed a projection map  $l : U_{\mathcal{O}}/U_{\mathcal{O}}^{(3)} \rightarrow \mathcal{H}_{2k+1}$  we get a homomorphism of the group  $C$  into  $Sp_{2k}$ . This is because  $Sp_{2k}$  may be identified with the group of automorphisms of  $\mathcal{H}_{2k+1}$  which restrict to the identity on the center, and because the action of  $C$  on  $U_{\mathcal{O}}$  will be given by such automorphisms.

Let  $\Theta_k^\psi$  denote the theta representation of  $\mathcal{H}_{2k+1} \rtimes \widetilde{Sp}_{2k}(\mathbb{A})$ . Then the above discussion indicates that the following integral

$$(8) \quad \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \theta_\phi^\psi(l(u)g) \varphi_\sigma(ug) du$$

is well defined and is left invariant under  $g \in C(F)$ . Here  $\theta_\phi^\psi$  is a vector in the space of  $\Theta_k^\psi$ . Depending on whether the group  $C(\mathbb{A})$  splits inside  $\widetilde{Sp}_{2k}(\mathbb{A})$ , integral (8) is a well defined function on either  $C(\mathbb{A})$  or the metaplectic double cover  $\widetilde{C}(\mathbb{A})$ . Notice the difference between integrals (6) and (8). Clearly the domain of integration is different. But more importantly, integral (8) will sometimes define an automorphic function on the group  $C(\mathbb{A})$ , and other times will define a genuine automorphic function on the metaplectic double cover of  $C(\mathbb{A})$ , whereas integral (6) always defines an automorphic function of  $C(\mathbb{A})$ . As we will work only with integrals of the type (8), this will be important for our construction.

The final case to consider is the case when the diagram which corresponds to the given unipotent orbit  $\mathcal{O}$  consists of zeros, ones and twos. In this case we combine the above two cases. In order to explain this, recall the group  $V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}$ . In the case when the weighted diagram has only zeros and twos, it is  $L_{\mathcal{O}}^{(1)}$ . In the case when there are only zeros and ones, it is  $L_{\mathcal{O}}^{(2)}$ . If both ones and twos are present, it contains a part of each of these groups. The group

$U_{\mathcal{O}}/V_{\mathcal{O}}^{(3)}$  is the direct product of the abelian group  $(V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}) \cap L_{\mathcal{O}}^{(1)}$ , and a generalized Heisenberg group with center  $(V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}) \cap L_{\mathcal{O}}^{(2)}$ . Define projections  $l : U_{\mathcal{O}} \rightarrow \mathcal{H}_{2n+1}$  and  $l' : U_{\mathcal{O}} \rightarrow (V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}) \cap L_{\mathcal{O}}^{(1)}$ . Here  $2n$  is the dimension of  $V_{\mathcal{O}}^{(1)}/V_{\mathcal{O}}^{(2)}$ .

As in [G-R-S1] equation (1.3), the corresponding Fourier coefficient is then given by

$$(9) \quad \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \theta_{\phi}^{\psi}(l(u)g) \varphi_{\sigma}(ug) \psi_{U_{\mathcal{O}}}(l'(u)) du$$

Here  $\theta_{\phi}^{\psi}$  is a vector in the theta representation defined on the suitable symplectic group. As before we have a character of  $V_{\mathcal{O}}^{(2)}(F) \backslash V_{\mathcal{O}}^{(2)}(\mathbb{A})$ , which, in this case is a combination of  $\psi_{U_{\mathcal{O}}}$  and the central character of  $\Theta_n^{\psi}$ , defined on the group  $(V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}) \cap L_{\mathcal{O}}^{(2)}$  using the projection  $l$ . This character corresponds to an element of the  $M_{\mathcal{O}}$ -module dual to  $V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}$ , and must be in the open orbit for the  $M_{\mathcal{O}}$ -action.

As an example, consider in  $H = F_4$  the unipotent class  $\mathcal{O} = B_2$ . Its diagram is given by

$$\overset{2}{0} - - - - 0 ==>== 0 - - - - \overset{1}{0}$$

As was mentioned above, the Levi part of  $P_{\mathcal{O}}$  is given by  $M_{\mathcal{O}} = Sp_4 \cdot GL_1^2$ . In this case, the action of  $M_{\mathcal{O}}$  on  $L_{\mathcal{O}}^{(1)}$  decomposes as a direct sum of two irreducible subrepresentations. One subrepresentation corresponds to the subgroup of  $M_{\mathcal{O}}$  on which  $T$  acts with the roots (1000), (1100), (1110), (1120), (1220), and the other corresponds to the subgroup of  $M_{\mathcal{O}}$  on which  $T$  acts with the roots (0001), (0011), (0111), (0121).

Similarly the action of  $M_{\mathcal{O}}$  on  $L_{\mathcal{O}}^{(2)}$  decomposes as a direct sum of two irreducible subrepresentations. One corresponds to a four-dimensional subgroup with roots, (1111), (1121), (1221), (1231), and the other corresponds to the subgroup  $U_{0122}$ . The function  $\sum_i n_i \alpha_i \mapsto \sum_i n_i c_i$  in this case is  $\sum_{i=1}^4 n_i \alpha_i \mapsto 2n_1 + n_4$ . The group  $V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)}$  consists of the first component of  $L_{\mathcal{O}}^{(1)}$  and the second component of  $L_{\mathcal{O}}^{(2)}$ . In other words, it corresponds to the six roots (1000), (1100), (1110), (1120), (1220) and (0122). The group  $U_{\mathcal{O}}/V_{\mathcal{O}}^{(3)}$  is the product of a five dimensional abelian group corresponding to the roots (1000), (1100), (1110), (1120), (1220) and a five dimensional Heisenberg group corresponding to the roots (0001), (0011), (0111), (0121), and (0122).

In order to define a Fourier coefficient in this case, we need to take a character in general position on the five dimensional abelian group, as well as a nontrivial character of the center of the Heisenberg group. We may then extend the latter to a theta representation.

Our five dimensional representation of  $M_{\mathcal{O}}$  can be thought of as the standard representation of  $SO_5$  (with  $Sp_4$  then appearing in its guise as  $Spin_5$ ). A point in this representation is in general position if the  $M_{\mathcal{O}}$ -invariant quadratic form does not vanish at that point. The character  $\psi_{U_{\mathcal{O}}}(u) = \psi(u_{1110})$  is easily seen to be in general position. The identity component

of its stabilizer in  $\text{Spin}_5$  is  $\text{Spin}_4 \cong SL_2 \times SL_2$ . The stabilizer in  $M_{\mathcal{O}}$  contains an additional one-dimensional torus, however, if we restrict to elements of  $M_{\mathcal{O}}$  which also act trivially on  $U_{0122}$ , the resulting group is  $SL_2^{\alpha_2} \cdot SL_2^{0120}$ , and indeed, the tables in [C] reflect that for this unipotent class, the group  $C$  is of type  $A_1 \times A_1$ . As mentioned earlier, the root subgroups  $U_{\alpha}$  for the roots  $(0001), (0011), (0111), (0121)$ , and  $(0122)$  form a subgroup isomorphic to  $\mathcal{H}_5$ . Thus, the corresponding Fourier coefficient, for this unipotent class is given by integral (9). Here  $\theta_{\phi}^{\psi}$  is a vector in the space of  $\Theta_2^{\psi}$ , the theta representation defined on  $\widetilde{Sp}_4(\mathbb{A})$ , and  $l$  is the projection from  $U_{\mathcal{O}}$  to  $\mathcal{H}_5$ . As in previous examples, the character  $\psi_{U_{\mathcal{O}}^{(2)}}$  which is nontrivial on the group  $U_{0122}$  is built in the theta function. We should mention, that since the group  $SL_2 \times SL_2$  does not split under the cover of  $\widetilde{Sp}_4(\mathbb{A})$ , it follows that the above integral defines a genuine automorphic function on the metaplectic cover  $\widetilde{SL}_2(\mathbb{A}) \times \widetilde{SL}_2(\mathbb{A})$ .

There is a third type of Fourier coefficient which can be attached to an orbit such that the weighted Dynkin diagram has ones in it in certain cases. To be specific: recall that  $U_{\mathcal{O}}/V_{\mathcal{O}}^{(2)}$  is essentially a symplectic vector space, equipped with a nondegenerate skew-symmetric form which is fixed by the group  $C$ . Let  $W$  be a maximal isotropic subspace, and let  $U_{\mathcal{O}}^{(3/2)}$  denote the preimage of  $W$  in  $U_{\mathcal{O}}$ . Let  $\psi_{U_{\mathcal{O}}^{(3/2)}}$  denote the trivial extension of  $\psi_{U_{\mathcal{O}}^{(2)}}$  to  $U_{\mathcal{O}}^{(3/2)}$ . Then the third type of Fourier coefficient is

$$(10) \quad \int_{U_{\mathcal{O}}^{(3/2)}(F) \backslash U_{\mathcal{O}}^{(3/2)}(\mathbb{A})} \varphi_{\sigma}(ug) \psi_{U_{\mathcal{O}}^{(3/2)}} du.$$

Note that this integral defines a function on  $C(F) \backslash C(\mathbb{A})$ , *only* if  $C$  normalizes  $U_{\mathcal{O}}^{(3/2)}$ , i.e., if the maximal isotropic subspace  $W \subset U_{\mathcal{O}}/V_{\mathcal{O}}^{(2)}$  is  $C$ -invariant.

A coefficient of this type was used in [B-G]. Also, the integral given in [J-S2] for the convolution  $L$ -function of  $GL_n \times GL_m$  involves a Fourier coefficient of this type whenever  $n - m$  is positive and even.

For completeness, we record the analogue of lemma 1.1 of [G-R-S1].

**Lemma 11.** Let  $f : \mathcal{H}_{2n+1}(F) \backslash \mathcal{H}_{2n+1}(\mathbb{A}) \rightarrow \mathbb{C}$  be any smooth function. The following are equivalent:

•

$$\int_{F \backslash \mathbb{A}} f((0|0|z)u) \psi(z) dz = 0, \quad (\forall u \in \mathcal{H}_{2n+1}(\mathbb{A})),$$

•

$$\int_{F \backslash \mathbb{A}} \int_{(F \backslash \mathbb{A})^n} f((0|y|z)u) \psi(z) dy dz = 0, \quad (\forall u \in \mathcal{H}_{2n+1}(\mathbb{A}))$$

•

$$\int_{\mathcal{H}_{2n+1}(F) \backslash \mathcal{H}_{2n+1}(\mathbb{A})} f(u'u) \theta_\phi^\psi(u') du' = 0, \quad (\forall u \in \mathcal{H}_{2n+1}(\mathbb{A}), \phi \in S(\mathbb{A}^n)).$$

*Proof.* The same arguments given to prove lemma 1.1 of [G-R-S1] actually prove this stronger statement.  $\square$

**Corollary 12.** For an automorphic representation  $(\sigma, V_\sigma)$  of  $H(\mathbb{A})$  (or a covering group), the following are equivalent:

- 1): integral (6) is zero for all  $\varphi_\sigma \in V_\sigma$ ,
- 2): integral (9) is zero for all  $\varphi_\sigma \in V_\sigma$  and all  $\phi \in S(\mathbb{A}^n)$ ,
- 3): integral (10) is zero for all  $\varphi_\sigma \in V_\sigma$ .

#### 4. Constructing Global Integrals

From the discussion in the previous section, it follows that given a unipotent orbit  $\mathcal{O}$  of an exceptional group  $H$ , we can associate with it a set of Fourier coefficients, which are defined on any automorphic representation of the group  $H(\mathbb{A})$ . In this section we shall use this to construct certain global integrals.

We start with the case where the corresponding diagram has only zeros and twos. Let  $\pi$  denote a cuspidal representation defined on the group  $C(\mathbb{A})$ . Given a maximal parabolic subgroup  $P$  of  $H$ , with Levi decomposition  $P = MU$ , we let  $E_\tau(h, s)$  denote an Eisenstein series of the group  $H(\mathbb{A})$  formed using an element of the induced representation  $\text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$ . Here  $\tau$  is any automorphic representation of the group  $M(\mathbb{A})$ .

The global integral we consider in this case is given by

$$(13) \quad \int_{C(F) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_\pi(g) E_\tau(ug, s) \psi_{U_{\mathcal{O}}}(u) du dg$$

In the case when the diagram corresponding to the unipotent orbit contains ones, we consider the integral

$$(14) \quad \int_{C(F) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_\pi(g) \theta_\phi^\psi(l(u)g) E_\tau(ug, s) \psi_{U_{\mathcal{O}}}(l'(u)) du dg$$

where  $l$  and  $l'$  project  $U_{\mathcal{O}}/V_{\mathcal{O}}^{(3)}$  onto its Heisenberg and abelian factors respectively. (See the discussion before equation (9).) As explained in the previous section, in this type of integral, the Fourier coefficient given by the integration along  $U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})$  may define a genuine automorphic function on  $\widetilde{C}(\mathbb{A})$ , the double cover of  $C(\mathbb{A})$ . If this is the case, then for integral (14) to be well defined, one of the representations  $\pi$  or  $E_\tau(g, s)$  should be defined on the double cover of the relevant group.

Both integrals (13) and (14) converge absolutely whenever  $s$  is not a pole of the Eisenstein series. This follows from the moderate growth property of automorphic forms and their Fourier coefficients, and the rapid decrease property of cusp forms. Also, each defines a meromorphic function on the whole complex plane, simply because the same is true of the Eisenstein series.

There are two other families of global integrals which are of interest. We refer to as reductive type integrals, and formally attach them to the zero nilpotent orbit. (For which  $C$  equals  $H$  itself.) The first reductive type are integrals of the form

$$(15) \quad \int_{H(F) \backslash H(\mathbb{A})} \varphi_\pi(h) \theta(h) E_\tau(h, s) dh$$

where  $\theta(h)$  is an element of some irreducible automorphic representation  $\Theta$  of  $H(\mathbb{A})$ . Here  $\pi$  denotes a cuspidal representation of the group  $H(\mathbb{A})$ . (The similarity in notation between  $\Theta$  and  $\Theta_n^\psi$  is deliberate: in practice  $\Theta$  is usually attached to a small orbit, and in this way is like  $\Theta_n^\psi$ , which is attached to the minimal orbit of  $Sp_{2n}$ .) The second family will be integrals of the type

$$(16) \quad \int_{Z(\mathbb{A})G(F) \backslash G(\mathbb{A})} \varphi_\pi(g) E_\tau(g, s) dg$$

where  $G$  is a reductive subgroup in  $H$ ,  $Z$  is its center, and  $\pi$  is a cuspidal representation defined on  $G(\mathbb{A})$ .

As mentioned before, when we deal with the groups  $E_6$  and  $E_7$ , we may need to use their similitude groups, in order to guarantee that the integral will unfold to a suitable model. In these cases, we may define the group  $C$  as the stabilizer of a suitable character  $\psi_{U_O}$  in the similitude group. It will certainly contain the center,  $Z$  of the similitude group, and we replace the integral over  $C(F) \backslash C(\mathbb{A})$  by an integral over  $Z(\mathbb{A})C(F) \backslash C(\mathbb{A})$ . Each of the above integrals converges absolutely, and satisfies a functional equation which is derived from the Eisenstein series.

Our main goal is to determine when these integrals are Eulerian. To try to answer this question, at least partly, we shall fix some notation. We need a good way to parameterize automorphic representations. One way to do it, which is suitable for our problem, is by means of Fourier coefficients.

In [G1], it is explained how to associate with each unipotent orbit a set of Fourier coefficients. This was essentially repeated in the previous section. The following definition is a way to attach to any automorphic representation a set of unipotent orbits.



**Definition 17.** ([G1] Definition 2.1) Let  $\sigma$  denote an automorphic representation of a given split reductive group  $G(\mathbb{A})$ , or of a metaplectic cover of such a group. We shall denote by  $\mathcal{O}rb_G(\sigma)$  the set of all unipotent orbits  $\mathcal{O}$  of  $G$  such that  $\sigma$  has a nonzero Fourier coefficient which corresponds to the unipotent orbit  $\mathcal{O}$ , and by  $\mathcal{O}_G(\sigma)$  the set of maximal elements of  $\mathcal{O}rb_G(\sigma)$ .

The structure of the set  $\mathcal{O}_G(\sigma)$  is not yet well understood. In [G1] there are several conjectures and assertions which are related to this set. To each representation which appears in integrals (13), (14), (15) and (16) we shall attach a number which we refer to as the Gelfand-Kirillov dimension of the representation. (By way of analogy with the “dimension” introduced in [K], theorem 2.4.1, (iib). Cf. also the references on p. 158 of [K] for related results on real Lie groups and the connection with the notion originally introduced by Gelfand and Kirillov.)

**Definition 18.** (1) Assume that the set  $\mathcal{O}_M(\tau)$  contains a single unipotent orbit  $\mathcal{O}$ . Let  $E_\tau(\cdot, s)$  denote the Eisenstein series defined right before integral (13). Then define the Gelfand-Kirillov dimension  $\dim \tau$  of  $\tau$  to be  $\frac{1}{2}\dim \mathcal{O}$ . For  $\text{Re } s$  large define the Gelfand-Kirillov dimension of  $E_\tau(\cdot, s)$  to be  $\dim U + \dim \tau$ .

(2) Since we will assume that  $\pi$  is generic, then we define  $\dim \pi = \dim U_{\max}^C$  where  $U_{\max}^C$  is the maximal unipotent subgroup of  $C$ .

(3) For the theta representation  $\Theta_n^\psi$  defined on  $\mathcal{H}_{2n+1}(\mathbb{A}) \rtimes \widetilde{Sp}_{2n}(\mathbb{A})$  we define  $\dim \Theta_n^\psi = n$ .

(4) Finally, for integrals of the type (15), we assume that  $\mathcal{O}_H(\Theta)$  is a singleton set  $\{\mathcal{O}\}$ , and define  $\dim \Theta = \frac{1}{2} \dim \mathcal{O}$ .

**Remark:** Observe that  $\mathcal{O}_C(\pi)$  is the singleton set containing the maximal orbit of  $C$ , which has dimension  $2 \dim U_{\max}^C$ . Also  $\mathcal{O}_{Sp_{2n}}(\Theta_n^\psi)$  is the singleton set containing the minimal orbit of  $Sp_{2n}$ , which has dimension  $2n$ . Finally, suppose that  $\mathcal{E}_{\tau,s}$  is the space of Eisenstein series attached to the induced representation  $\text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$ . Then, following [G1], proposition 5.16, we expect that

$$\mathcal{O}_M(\tau) = \{\mathcal{O}\} \implies \mathcal{O}_H(\mathcal{E}_\tau) = \{\mathcal{O}'\}, \text{ where } \dim \mathcal{O}' = \dim \mathcal{O} + 2 \dim U(P).$$

Actually, we can prove it for low rank groups, including the exceptional group  $F_4$ . However, since we will not need it, we omit it.

Note that the representation  $\tau$ , or the representation  $\Theta$  appearing in (15) could itself be an Eisenstein series. In this case the dimension is defined using the unique element of  $\mathcal{O}_M(\tau)$  or  $\mathcal{O}_H(\Theta)$ , and not using definition 17. The motivation for this is that in the practice of unfolding one needs to know information regarding what Fourier coefficients

these representations support, and prefers, where possible, to make arguments that are valid for any representation supporting a given set of Fourier coefficients.

Next we consider the cuspidal representation  $\pi$  which appears in integrals (13) and (14). Even though our main interest is the case when  $\pi$  is generic, and hence  $\dim \pi$  is well defined, for possible future applications we prove

**Lemma 19.** Assume that  $H = F_4$  or  $H = E_6$ . For any nontrivial unipotent orbit of  $H$ , let  $\pi$  be an irreducible cuspidal representation of  $C(\mathbb{A})$ . Then  $\dim \pi$  is well defined.

*Proof.* The lemma is proved by a case by case consideration. The type of  $C$  for each orbit is given in the tables of [C]. It will be a group of type  $A_1, A_2, C_2, G_2, A_3, B_3, C_3, A_5$ , or some combination of these.

If  $G$  is a simple group of any of the types listed above, except for  $C_3$  and  $A_5$ , then the orbits of  $G$  are totally ordered, and so every subset has a unique maximal element. For  $C_3$ , the set of all special orbits is totally ordered, and it is proved in [G-R-S1] theorem 2.1 that any orbit appearing in  $\mathcal{O}_C(\pi)$  must be special. (For a definition of “special,” see [C], p. 389. For an alternate description for classical groups, see [C-M], p. 100.)

This leaves a single orbit in  $E_6$  when the stabilizer is of type  $A_5$ . In fact (since we work with the split form of  $E_6$ ) the stabilizer for this case is isomorphic to  $SL_6$  (If we work in  $GE_6$ , the stabilizer is isomorphic to  $\{g \in GL_6 : \det g \text{ is a square}\}$ . This case was considered in [G-H1].) Every cuspidal representation of  $SL_6$  is generic with respect to some generic character, so in this case  $\dim \pi$  is always defined and equal to the dimension of the maximal unipotent subgroup in  $SL_6$ .  $\square$

Finally, we consider the representations occur in integrals (15) and (16). In these cases, we will assume that  $\pi$  is generic, and for the function  $\theta$  which occurs in integral (15), we will assume that its dimension is well defined. This will not cause a problem, since for the cases we will consider this assumption will be satisfied automatically. We record all this in the next lemma.

**Lemma 20.** With the above assumptions, the notion of dimension is well defined for any representation occurring in integrals (13), (14), (15) and (16).

Recall that our goal is to study the question when are the integrals (13)-(16) are Eulerian. Following the discussion in the introduction of [G3] we introduce the following definition.

**Definition 21. (The basic equation:)** An integral of the type (13), (14), (15), or (16), is said to satisfy the basic equation if the sum of the Gelfand-Kirillov dimensions of the representations involved in the integral is equal to the dimension of the domain over which we integrate.

We now write the basic equation in detail. Consider first integral (13). The basic equation in this case is given by

$$(22) \quad \dim C + \dim U_{\mathcal{O}} = \dim \pi + \dim E_{\tau}(\cdot, s)$$

For integral (14) the equation is given by

$$\dim C + \dim U_{\mathcal{O}} = \dim \pi + \dim E_{\tau}(\cdot, s) + \dim \Theta_k^{\psi}.$$

Here  $k = \frac{1}{2}(\dim U_{\mathcal{O}} - \dim V_{\mathcal{O}}^{(2)})$  is the unique integer such that  $U_{\mathcal{O}}$  has a projection onto  $\mathcal{H}_{2k+1}$ , with the kernel of this projection contained in  $V_{\mathcal{O}}^{(2)}$ . In the first identity above, we have  $\dim U_{\mathcal{O}} = \frac{1}{2} \dim \mathcal{O}$ , while in the second case, we have  $\dim U_{\mathcal{O}} - \dim \Theta_k^{\psi} = \frac{1}{2} \dim \mathcal{O}$ . This follows from lemma 4.1.1, p. 56 of [C-M].

Hence, it follows that if we start from a nonzero unipotent orbit  $\mathcal{O}$  then the relevant integral (13) or (14) satisfies the basic equation if and only if it satisfies the equation

$$\dim C + \frac{1}{2} \dim \mathcal{O} = \dim \pi + \dim E_{\tau}(\cdot, s)$$

Since we assumed that  $\pi$  is generic, it follows that  $\dim \pi = \dim U_C$  where  $U_C$  is the maximal unipotent subgroup of  $C$ . From definition 18 we also have that  $\dim E_{\tau}(\cdot, s) = \dim U(P) + \dim \tau$ . Plugging this inside the above equation we obtain the basic equation

$$(23) \quad \dim \tau = \dim B_C + \frac{1}{2} \dim \mathcal{O} - \dim U(P)$$

where  $B_C$  is the Borel subgroup of  $C$ . Notice that this last formula holds also for integral (16), if we take  $\dim \mathcal{O} = 0$  and  $C = G$ . Arguing in a similar way for integral (15), we deduce that the formula that should hold in this case is

$$(24) \quad \dim \theta + \dim \tau = \dim B_H - \dim U(P)$$

where  $B_H$  is the Borel subgroup of  $H$ .

We consider a few examples. We start with an example related to integral (15) for the group  $H = F_4$ . In this case, equation (24) reduces to  $\dim \theta + \dim \tau = 28 - \dim U(P)$ . Since  $F_4$  has two maximal parabolic subgroups  $P$  such that  $\dim U(P) = 20$ , and two such that  $\dim U(P) = 15$ , it follows that there are two possibilities. First suppose that  $\dim U(P) = 20$ . This is the case if  $P = P_2$  or  $P_3$ . Then (24) reduces to  $\dim \theta + \dim \tau = 8$ . Now,  $\dim \theta$  is equal to one half of the dimension of a nontrivial nilpotent orbit of  $F_4$ . The dimensions of the orbits are listed on page 128 of [C-M]. The minimal orbit has dimension 16, so it follows that in this case  $\theta$  must be attached to the minimal orbit, and  $\tau$  must be a character. An example of a representation which is attached to the minimal orbit of  $F_4$  is constructed and studied in [G4]. In fact, it is a representation which is defined on the double cover of  $F_4(\mathbb{A})$ .

Now suppose that  $\dim U(P) = 15$ . This is the case when  $P = P_1$  or  $P = P_4$ , so that the Levi part is isomorphic to  $GS p_6$  or  $GSpin_7$ , respectively. In this case, (24) reduces to

$\dim \theta + \dim \tau = 13$ . It then follows from [C-M] page 128, that either  $\theta$  is attached to the minimal orbit, and  $\dim \theta = 8$ , or else  $\theta$  is attached to the unipotent orbit  $\tilde{A}_1$ , and  $\dim \theta = 11$ . If  $\dim \theta$  is 11, then we need  $\dim \tau = 2$ . However, neither  $GS p_6$  nor  $GSpin_7$ , has an orbit whose dimension is 4. So, once again  $\theta$  must be attached to the minimal orbit. The basic equation will be satisfied if we can find a representation  $\tau$  defined on  $GS p_6$  or  $GSpin_7$ , such that the dimension is five. Each of these two groups has a unique unipotent orbit  $\mathcal{O}$  of dimension 10. The unique 10-dimensional unipotent orbit of  $GS p_6$  is  $(2^2 1^2)$ , and that of  $GSpin_7$  is  $(31^5)$ . (For the parametrization of orbits by partitions, see [C-M], p. 70. To compute the dimension, use [C-M], §5.2, and lemma 4.1.1.)

From this, it follows that there are essentially six different possibilities for an integral of the form (15). First, suppose that the Eisenstein series  $E_\tau(h, s)$  comes from a representation induced from either  $P_1$  or  $P_4$ , and  $\tau$  is any representation of  $GS p_6$  or  $GSpin_7$  whose dimension is five. Since the representation  $\theta$  is on the double cover, it follows that either  $\pi$  or  $E_\tau(g, s)$  has to be defined on the double cover. If we form the Eisenstein series on the double cover of  $H$ , we obtain an integral of the form

$$\int_{H(F) \backslash H(\mathbb{A})} \varphi_\pi(h) \tilde{\theta}(h) \tilde{E}_\tau(h, s) dh.$$

Here  $\tau$  is an automorphic representation defined on the double cover of  $GS p_6(\mathbb{A})$  or  $GSpin_7(\mathbb{A})$  whose dimension is five. A second option is to let  $\pi$  be a cuspidal representation defined on the metaplectic double cover  $\tilde{H}(\mathbb{A})$  of  $H(\mathbb{A})$ . Thus, in this case, we have

$$\int_{H(F) \backslash H(\mathbb{A})} \tilde{\varphi}_\pi(h) \tilde{\theta}(h) E_\tau(h, s) dh.$$

In this option  $\tau$  is an automorphic representation of  $GS p_6(\mathbb{A})$  or  $GSpin_7(\mathbb{A})$  whose dimension is five.

Now suppose that the Eisenstein series  $E_\tau(h, s)$  is attached to the parabolic subgroup  $P_2$  or  $P_3$ . In this case  $\tau$  must be a character, and it follows that  $E_\tau(h, s)$  can not be a genuine function on the double cover of  $H(\mathbb{A})$ . Hence we must assume that  $\pi$  is a cuspidal representation defined on the double cover of  $H(\mathbb{A})$ . The integral we construct is similar to the above, where the only change is the choice of the Eisenstein series.

As a second example, consider the unipotent orbit  $\mathcal{O} = A_1$  for the group  $H = E_6$ . Its diagram is given in (5). The relevant equation to consider is (23). It follows from [C-M] that  $\frac{1}{2} \dim \mathcal{O} = 11$ , and from [C] page 402, it follows that  $C$  is of type  $A_5$ . Thus  $\dim B_C = 20$ . Hence equation (23) is  $\dim \tau = 31 - \dim U(P)$ . The group  $E_6$  has 4 non associated maximal standard parabolic subgroups. Their unipotent radicals have dimensions  $\dim U(P_1) = 16, \dim U(P_2) = 21, \dim U(P_3) = 25$  and  $\dim U(P_4) = 29$ , respectively. The

integral to consider in this case is the one written in (14). We consider each parabolic in turn. First, if  $P = P_1$ , then  $\dim \tau = 15$ . The Levi part of  $P_1$  is a group of type  $D_5$ . It is not hard to check that this group has a unique unipotent orbit such that half of its dimension is 15. This unipotent orbit is  $(3^3 1)$ . Next, when  $P = P_2$ , we obtain  $\dim \tau = 10$ . Since the Levi part of  $P_2$  is a group of type  $A_5$ , and since this group has no unipotent orbit of dimension twenty, it follows that the set of solutions to equation (23) is empty in this case. For  $P = P_3$  we obtain  $\dim \tau = 6$ . The Levi part of this parabolic subgroup is a group of type  $A_1 \times A_4$ . There is one possibility. On the group of type  $A_1$  we take a representation attached to the orbit  $(1^2)$ , i.e., a character, and, on the group of type  $A_4$ , we take a representation attached to the unipotent orbit  $(2^2 1)$ . We denote this by  $(1^2 | 2^2 1)$ . Finally, for  $P = P_4$ , we have  $\dim \tau = 2$  and since the Levi part of this parabolic subgroup is of type  $A_2 \times A_1 \times A_2$ , then the only choice is take a representation attached to the unipotent orbit  $(2 1)$  on one of the two  $A_2$ -components, and take characters on the other two components. By symmetry, it does not matter which of the two  $A_2$ -components we take the representation attached to  $(2 1)$  on, so in this case there is only one possibility, which we represent by  $(2 1 | 1^2 | 1^3)$ .

In the following table we summarize the following data for the group  $H = F_4$ . First, in the leftmost column we label the unipotent orbit in question. The non special orbits are labeled with bold letters. In the next column we write the type of the group  $C$ . Since  $H = F_4$ , we have four maximal parabolic subgroups. In the next two columns we write down the dimensions of  $\tau$  required so that equation (23) holds. Since the unipotent radicals of  $P_1$  and  $P_4$  have the same dimension, we can write them in the same column. Similarly for  $P_2$  and  $P_3$ . In the next four columns we write down possible unipotent orbits, defined on the Levi part of  $P$ , whose dimension is twice the dimension of  $\tau$ . For example, consider the unipotent orbit  $B_2$ . Since it is not special we label it with bold letters. The stabilizer is of type  $A_1 + A_1$ . When  $P$  is such that its Levi part is of type  $C_3$  or  $B_3$ , then the dimension of  $\tau$  required to satisfy equation (23) is seven. The group of type  $C_3$  has 2 unipotent orbits of dimension 14. They are  $(3^2)$  and  $(4 1^2)$ . They are listed at the fifth column. Notice, that since  $(4 1^2)$  is not special we labeled it with bold letters.

Finally, the first row in the table is devoted to the integral of type (16). The numbers indicated are the numbers  $\dim \tau + \dim \theta$  such that equation (24) is satisfied. For example, if  $P = P_1$  then  $\dim \tau + \dim \theta$  must equal 13, and, as discussed above, this means that  $\theta$  must be attached to the minimal orbit of  $F_4$ . The Bala-Carter label of this orbit is  $\mathbf{A}_1$ . Also,  $\tau$  must be attached to the orbit  $(2^2 1^2)$ .

One may also consider what we refer as the “dual integrals” to integrals (9) and (10), in which the roles of the cusp form and the Eisenstein series are reversed. For example, the

integral dual to integral (9) is given by

$$\int_{C(F)\backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}(F)\backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_{\pi}(ug) E_{\tau}(g, s) \psi_{U_{\mathcal{O}}}(u) du dg,$$

where  $\varphi_{\pi}$  is a cusp form in the space of a cuspidal automorphic representation of  $H(F)\backslash H(\mathbb{A})$  and  $E_{\tau}$  is an Eisenstein series on  $C(F)\backslash C(\mathbb{A})$ . Convergence of such integrals follows from the fact that the Fourier coefficients of cusp forms inherit their rapid decay property, as has recently been established in general by Miller and Schmid [MS]. However, in this paper we are mainly interested in integrals such that the representation  $\pi$  is generic, and which satisfy the equation

$$\dim C + \frac{1}{2} \dim \mathcal{O} = \dim \pi + \dim E_{\tau}(\cdot, s)$$

Since we require that  $\pi$  is generic, then  $\dim \pi = 24$ . Further,  $\dim E_{\tau}(\cdot, s)$  is strictly positive. To be precise, it is at least equal to the smallest integer which is the dimension of the unipotent radical of a parabolic of  $C$ . There are only three nonzero orbits in  $F_4$  such that  $\frac{1}{2} \dim \mathcal{O} + \dim C - 24$  is positive. These are  $\mathbf{A}_1$ ,  $\tilde{A}_1$ , and  $\tilde{A}_2$ . In the case of  $\tilde{A}_1$ , we have  $\frac{1}{2} \dim \mathcal{O} + \dim C - 24 = 2$ , and  $C \cong SL_4$ . This implies  $\dim E_{\tau}(\cdot, s) \geq 3$ , ruling out this case.

For  $\tilde{A}_2$  we have  $\frac{1}{2} \dim \mathcal{O} + \dim C - 24 = 5$ , and  $C \cong G_2$ . If  $P$  is either of the maximal parabolic subgroups of  $G_2$ , then the dimension of the unipotent radical of  $P$  is 5, and so  $\dim E_{\tau}(\cdot, s) = 5$  if  $\tau$  is a character of the Levi part of  $P$ . One of these cases can be reduced to a Whittaker integral, and appears in [G-R].

For  $\mathbf{A}_1$ , we have  $\frac{1}{2} \dim \mathcal{O} + \dim C - 24 = 5$ , and  $C \cong Sp_6$ . It is possible to arrange that  $\dim E_{\tau}(\cdot, s) = 5$  only by taking  $E_{\tau}(\cdot, s)$  to be an Eisenstein series attached to the induced representations  $\text{Ind}_{P(\mathbb{A})}^{Sp_6(\mathbb{A})} \tau \delta_P^s$ , where  $P$  the Levi subgroup of  $P$  is isomorphic to  $GL_1 \times Sp_4$ , and  $\tau$  is a character.

Thus, there are in effect only three cases for this type of integral, and one of them has already appeared in the literature. We remark that a preliminary analysis suggests that neither of the others can unfold to a Whittaker integral.

**The group  $F_4$**

label	stabilizer	$P_1, P_4$	$P_2, P_3$	$C_3$	$B_3$	$A_2 + A_1$	$A_1 + A_2$
1	$F_4$	13	8	$[\mathbf{A}_1, (2^2 1^2)]$	$[\mathbf{A}_1, (31^4)]$	$[\mathbf{A}_1, (1^3 1^2)]$	$[\mathbf{A}_1, (1^2 1^3)]$
$\mathbf{A}_1$	$C_3$	5	0	$(2^2 1^2)$	$(31^4)$	$(1^3 1^2)$	$(1^2 1^3)$
$\tilde{A}_1$	$A_3$	5	0	$(2^2 1^2)$	$(31^4)$	$(1^3 1^2)$	$(1^2 1^3)$
$A_1 + \tilde{A}_1$	$A_1 + A_1$	1	—	—	—	—	—
$A_2$	$A_2$	5	0	$(2^2 1^2)$	$(31^4)$	$(1^3 1^2)$	$(1^2 1^3)$
$\tilde{A}_2$	$G_2$	8	3	$(42)$	$(51^2)$	$(3 1^2), (21 2)$	$(1^2 3), (2 21)$
$\mathbf{A}_2 + \tilde{\mathbf{A}}_1$	$A_1$	4	—	—	$(\mathbf{2}^2 \mathbf{1}^3)$	—	—
$\mathbf{B}_2$	$A_1 + A_1$	7	2	$(3^2)$ $(\mathbf{41}^2)$	$(3^2 1)$	$(21 1^2)$	$(1^2 21)$
$\tilde{\mathbf{A}}_2 + \mathbf{A}_1$	$A_1$	5	0	$(2^2 1^2)$	$(31^4)$	$(1^3 1^2)$	$(1^2 1^3)$
$\mathbf{C}_3(\mathbf{a}_1)$	$A_1$	6	1	$(2^3)$	$(32^2)$	$(1^3 2)$	$(2 1^3)$
$F_4(a_3)$	0	5	0	$(2^2 1^2)$	$(31^4)$	$(1^3 1^2)$	$(1^2 1^3)$
$B_3$	$A_1$	8	3	$(42)$	$(51^2)$	$(3 1^2), (21 2)$	$(1^2 3), (2 21)$
$C_3$	$A_1$	8	3	$(42)$	$(51^2)$	$(3 1^2), (21 2)$	$(1^2 3), (2 21)$
$F_4(a_2)$	0	7	2	$(3^2)$ $(\mathbf{41}^2)$	$(3^2 1)$	$(21 1^2)$	$(1^2 21)$
$F_4(a_1)$	0	8	3	$(42)$	$(51^2)$	$(3 1^2), (21 2)$	$(1^2 3), (2 21)$
$F_4$	0	9	4	$(6)$ LS	$(7)$ LS	$(3 2)$ LS	$(2 3)$ LS

## 5. Unfolding Global integrals

In this section we indicate several steps in the unfolding of integrals (13) and (14). The integrals (15) and (16) are treated similarly. Integral (14) is more complicated since it also involves the theta representation. However, at least the first steps are similar, hence we will first concentrate on integral (13), and then discuss integral (14) more briefly. We will also return to integral (14) in section 8.

**5.1. The General Process.** The purpose of the unfolding process is to determine if the integral is Eulerian, by reducing things to certain functionals defined on the representations in question. Then one studies if these functionals are unique. For example, integral (13) is given by

$$(25) \quad I = \int_{C(F) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_{\pi}(g) E_{\tau}(ug, s) \psi_{U_{\mathcal{O}}}(u) du dg$$

The two representations which are of import to us are  $\pi$  and  $\tau$ . After we finish the unfolding process we will obtain certain inner integration, which may be an integral over a reductive group, or a Fourier coefficient or some combination of the two. Since we assume that the cuspidal representation  $\pi$  is generic, it is natural to hope to obtain a Whittaker coefficient after the unfolding process is finished. We assume that  $\tau$  has Gelfand-Kirillov dimension as specified by the dimension formula. In most cases, this means that  $\tau$  is not generic.

We now write several steps in the unfolding process. Unfolding the Eisenstein series we obtain

$$\begin{aligned} E_{\tau}(ug, s) &= \sum_{\gamma \in P(F) \backslash H(F)} f_{\tau}(\gamma ug, s) = \\ &\sum_{w \in P(F) \backslash H(F) / P_{\mathcal{O}}(F)} \sum_{\gamma \in (w^{-1}P(F)w \cap P_{\mathcal{O}}(F)) \backslash P_{\mathcal{O}}(F)} f_{\tau}(w\gamma ug, s) \end{aligned}$$

The first sum is finite, and representatives can be chosen to be Weyl elements of  $H$ . Thus, if we define

$$I_w = \int_{C(F) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_{\pi}(g) \sum_{\gamma \in (w^{-1}P(F)w \cap P_{\mathcal{O}}(F)) \backslash P_{\mathcal{O}}(F)} f_{\tau}(w\gamma ug, s) \psi_{U_{\mathcal{O}}}(u) du dg$$

then  $I = \sum_w I_w$  where the sum is over the space of double cosets  $P(F) \backslash H(F) / P_{\mathcal{O}}(F)$ . Moreover, we can write the sum

$$\sum_{\gamma \in (w^{-1}P(F)w \cap P_{\mathcal{O}}(F)) \backslash P_{\mathcal{O}}(F)} = \sum_{\gamma \in Q_w(F) \backslash M_{\mathcal{O}}(F)} \sum_{\mu \in U_{\mathcal{O}}^w(F) \backslash U_{\mathcal{O}}(F)} = \sum_{\gamma \in Q_w(F) \backslash M_{\mathcal{O}}(F)} \sum_{\mu \in U_{\mathcal{O},w}(F)}$$



Here  $Q_w = M_{\mathcal{O}} \cap w^{-1}Pw$ , which is a parabolic subgroup of  $M_{\mathcal{O}}$ ,  $U_{\mathcal{O}}^w = U_{\mathcal{O}} \cap w^{-1}Pw$ , and  $U_{\mathcal{O},w} = U_{\mathcal{O}} \cap w^{-1}U^-w$ , where  $U^-$  is the unipotent radical of the parabolic subgroup which is opposite to  $P$ .

The next step is to consider the space  $Q_w(F) \backslash M_{\mathcal{O}}(F) / C(F)$ . We have

$$\sum_{\gamma \in Q_w(F) \backslash M_{\mathcal{O}}(F)} = \sum_{\nu \in Q_w(F) \backslash M_{\mathcal{O}}(F) / C(F)} \sum_{\gamma \in (\nu^{-1}Q_w(F)\nu \cap C(F)) \backslash C(F)}$$

Plugging all this inside  $I_w$ , we then collapse summation with integration. Thus, if we denote

$$(26) \quad I_{w,\nu} = \int_{(\nu^{-1}Q_w(F)\nu \cap C(F)) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}^{w\nu}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_{\pi}(g) f_{\tau}(w\nu u g, s) \psi_{U_{\mathcal{O}}}(u) du dg$$

then  $I_w = \sum_{\nu} I_{w,\nu}$ , where the sum is over representatives  $\nu$  for the set  $Q_w(F) \backslash M_{\mathcal{O}}(F) / C(F)$ , and  $U_{\mathcal{O}}^{w\nu} = U_{\mathcal{O}} \cap (w\nu)^{-1}Pw\nu$ .

Denote  $L_{\nu} = \nu^{-1}Q_w\nu \cap C$  and  $V = (w\nu)U_{\mathcal{O}}^{w\nu}(w\nu)^{-1}$ . For  $v \in V$  we denote  $\psi_V(v) = \psi_{U_{\mathcal{O}}}((w\nu)^{-1}vw\nu)$ . Factoring the measure in the above integral,  $I_{w,\nu}$  is equal to

$$(27) \quad \int_{L_{\nu}(\mathbb{A}) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}^w(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} \int_{L_{\nu}(F) \backslash L_{\nu}(\mathbb{A})} \int_{V(F) \backslash V(\mathbb{A})} F_{\tau}(m(v)m(w\nu l(w\nu)^{-1}); w\nu u g, s) \\ \psi_V(v) \varphi_{\pi}(lg) \psi_{U_{\mathcal{O}}}(u) |\delta_P(w\nu l(w\nu)^{-1})|^s dv dl du dg$$

To explain the notation, we recall that the natural definition for an element of  $\text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$  is as a function from  $H(\mathbb{A})$  taking values in the space of  $\tau \otimes \delta_P^s$ . Since the space of  $\tau \otimes \delta_P^s$  consists of automorphic forms:  $M(\mathbb{A}) \rightarrow \mathbb{C}$ , an element of  $\text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$  thus becomes a function  $M(\mathbb{A}) \times H(\mathbb{A}) \rightarrow \mathbb{C}$ , which we write  $F_{\tau}(m; h, s)$ . This function satisfies

$$F_{\tau}(m_1; m_2 h, s) = \tau \otimes \delta_P^s(m_2) F_{\tau}(m_1; h, s) = F_{\tau}(m_1 m_2; h, s).$$

for all  $h \in H(\mathbb{A})$ ,  $m_1, m_2 \in M(\mathbb{A})$  and  $s \in \mathbb{C}$ . In order to define Eisenstein series, one passes to an alternate realization of  $\text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$  consisting of complex-valued functions, by defining  $f_{\tau}(h, s) = F_{\tau}(e; h, s)$ . See page 60 of [G-PS-R] for a further discussion. Also, in (27) we have denoted the natural projection from the parabolic subgroup  $P$  to its Levi  $M$  by  $m$ .

To proceed with the unfolding process we consider the following integral

$$(28) \quad \int_{L_{\nu}(F) \backslash L_{\nu}(\mathbb{A})} \int_{V(F) \backslash V(\mathbb{A})} \varphi_{\pi}(lg) \varphi_{\tau}(m(v)m(w\nu l\nu^{-1}w^{-1})) \psi_V(v) dv dl$$

It follows from the discussion after integral (27) that, as far as the unfolding process goes, unfolding integral (28) is equivalent to unfolding integral (27). We emphasize that integral (28) is not an inner integration to integral (27).

Thus, we have reduced the computation to a lower rank situation.

We expect that for most elements  $\nu$ , integral (27) will be zero. We are interested in those cases where the following holds.

**Definition 29.** We will say that the global integral (25) is an **open orbit type** integral if there is a unique element  $w_0 \in P(F) \backslash H(F) / P_{\mathcal{O}}(F)$  and a unique element in the space  $\nu_0 \in Q_w(F) \backslash M_{\mathcal{O}}(F) / C(F)$  such that the integral (28) satisfies the dimension identity

$$(30) \quad \dim \pi + \dim \tau = \dim L_{\nu_0} + \dim V$$

If (30) is satisfied by more than one element  $\nu$ , but we can show that  $I_{w,\nu}$  vanishes for all but one, then we shall say that the integral (25) is **weakly open orbit type**.

To explain this terminology, consider the action of  $P \times U_{\mathcal{O}}C$  on  $H$  by  $(p, ug) \cdot h = ph(ug)^{-1}$ . Then the stabilizer of  $w\nu$  is isomorphic to  $L_{\nu}U_{\mathcal{O}}^{w\nu}$ , and in particular is of dimension  $\dim L_{\nu} + \dim V$ . The identity (22) implies that

$$\dim \pi + \dim \tau = \dim U_{\mathcal{O}} + \dim C + \dim P - \dim H.$$

This is equal to  $\dim L_{\nu} + \dim V$  if and only if the orbit of  $w\nu$  has the same dimension as  $H$ , in which case it is open. Observe that an open orbit for this action (i.e., an open  $P, U_{\mathcal{O}}C$ -double coset), will certainly be contained in the open  $P, P_{\mathcal{O}}$ -double coset. Let  $w_0$  denote a representative for this orbit, chosen to be the shortest element of the Weyl group which is contained in the orbit. Then the mapping  $Q_{w_0}\nu C \rightarrow Pw_0\nu CU_{\mathcal{O}}$  is a well defined bijection from  $Q_{w_0} \backslash M / C$  to the space of  $P, U_{\mathcal{O}}C$ -double cosets contained in the open  $P, P_{\mathcal{O}}$ -double coset. Hence, there is an open  $P, U_{\mathcal{O}}C$ -double coset in  $H$  if and only if there is a  $Q_{w_0}, C$ -double coset in  $M_{\mathcal{O}}$ .

Now assume that  $w\nu \in H(F)$ . (Recall that we identify  $H$  with  $H(\overline{F})$ , where  $\overline{F}$  is a fixed algebraic closure of  $F$ , and likewise for other algebraic groups.) Then  $H(F) \cap Pw\nu U_{\mathcal{O}}C$  may not be a single  $P(F) \times U_{\mathcal{O}}(F)C(F)$ -orbit. In cases where it is not, the integral (25) is not of open orbit type, but it may still be of weakly open orbit type.

Notice that the dimension identity given in definition 29 is analogous to the dimension identity (22), but this time the representations under consideration are  $\pi$  and  $\tau$  (as opposed to  $\pi$  and  $\text{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \tau \otimes \delta_P^s$ ). Experience indicates the following:

**Conjecture 31.** Suppose that integral (25) is an open orbit type integral. Then integral (27) is zero, except when  $w = w_0$  and  $\nu = \nu_0$ . Moreover, the element  $w_0$  can be chosen to be Weyl element which corresponds to the longest Weyl element in the space  $P(F) \backslash H(F) / P_{\mathcal{O}}(F)$ .

At this point our ability to check whether Conjecture 31 holds, is done only by carrying out the unfolding process. In addition, unfolding is, in general, necessary to determine whether

or not an integral is weakly open orbit type, in the case when  $Q_{w_0} \backslash M_{\mathcal{O}}/C$  possesses an open orbit, which is a union of two or more orbits for the action of  $Q_{w_0}(F) \times C(F)$ .

We mention that all of the above is similar for integrals of the form (14). The theta function which appears in the integral will appear in the same way in the analogue of integral (28), and definition 29 and conjecture 31 are similar with the addition of the theta function. See section 8.3 for some details.

We make one more definition.

**Definition 32.** Suppose that conjecture 31 holds. We will refer to the integral (25) as a **preWhittaker** integral if, for any representation  $\tau$  such that  $\mathcal{O}(\tau)$  is given in the above tables, there is a process of further unfolding of integral (28) which involves the Whittaker coefficient of the representation  $\pi$ . We shall also refer to an integral as **weakly preWhittaker** if we are able to unfold the integral, but only by placing additional conditions on  $\tau$  (e.g., requiring that  $\tau$  be an Eisenstein series) which do not follow from the conditions on  $\mathcal{O}(\tau)$ .

To prove that a given global integral is weakly preWhittaker, we only need to find a representation  $\tau$  such that integral 28 will unfold to a Whittaker coefficient in  $\pi$ . This is frequently quite easy. Proving that a global integral is preWhittaker is more challenging, and generally requires some knowledge about the class of all representations of the Levi  $M$  attached to a given orbit. It is not clear how to prove that an integral is not weakly preWhittaker, or preWhittaker. Even though it is usually easy to guess when an integral does not unfold to a Whittaker coefficient in  $\pi$ , it is not clear how to prove it.

To satisfy the dimension equations, we require that  $\tau$  be a representation which supports certain Fourier coefficients corresponding to specific unipotent orbits. This is a method of sorting automorphic representations into classes which is convenient for our purposes. Another method is to sort the representations into cuspidal representations, Eisenstein series, and residual representations, and then sort the Eisenstein series according to which parabolic subgroup they are induced from, etc. Note that knowing the Gelfand-Kirillov dimension of a representation  $\tau$ , or even the more refined invariant  $\mathcal{O}_G(\tau)$ , does not tell us where  $\tau$  falls in this other classification. For example, a representation which is generic could be cuspidal, or it could be an Eisenstein series induced from any parabolic subgroup. However, it may happen that for some representation  $\tau$  integral (28) can be further reduced by means of Fourier expansions to a Whittaker type integral. See the examples given in sections 7 and 9.

There is one exception to the general principle that  $\mathcal{O}_G(\tau)$  does not determine the type of  $\tau$ .

**Lemma 33.** If  $\mathcal{O}_G(\tau)$  is the singleton set containing the zero orbit, then  $\tau$  is a character.

*Proof.* One may express  $G$  as a quotient of  $G_1 \times \cdots \times G_k \times T$  for some simple simply connected groups  $G_1, \dots, G_k$  and torus  $T$ , so the general case reduces to the case when  $G$  is simple and simply connected.

We show that if  $\varphi$  is an automorphic form which does not support any Fourier coefficient attached to a unipotent orbit then  $\varphi$  is invariant by  $U_\alpha(\mathbb{A})$  for all  $\alpha$ , and hence invariant by  $(G, G)(\mathbb{A})$ , where  $(G, G)$  is the commutator subgroup.

To prove  $U_\alpha(\mathbb{A})$ -invariance for all  $\alpha$ , it suffices to prove it for one element of each Weyl-orbit in  $\Phi(G, T)$ , i.e., for one root of each length. If  $\mathcal{O}_{\min}$  is the minimal orbit then  $U_{\mathcal{O}_{\min}}$  is a Heisenberg group with center  $U_\alpha$  for some long root  $\alpha_L$ . The fact that Fourier coefficients attached to this orbit vanish implies that  $\varphi$  is invariant by the center. This by itself treats the case when  $G$  is simply laced. In the case when there are long and short roots, this proves that  $\varphi$  is invariant by  $U_\alpha(\mathbb{A})$  for all long roots  $\alpha$ . Further, in each case one may identify a second orbit  $\mathcal{O}'$  such that  $V_{\mathcal{O}'}^{(2)}$  is the product of  $U_{\alpha_S}$  for some unique short root  $S$ , and possibly one or more root subgroups corresponding to long roots. Further, in each case the character  $\psi_{V_{\mathcal{O}'}^{(2)}}(v) = \psi(v_{\alpha_S})$  is in general position. Since  $\varphi$  does not support any Fourier coefficients attached to  $\mathcal{O}'$  either, it follows that  $\varphi$  is invariant by  $U_\alpha(\mathbb{A})$  for short roots as well.  $\square$

**5.2. A Special Case.** In this subsection we consider a special case which is valid for any group  $H$ . We explain the details for the case of integral (13), but it is clearly the same for integrals of the type given by (14). Let  $\mathcal{O}$  be such that  $P_{\mathcal{O}}$  is maximal. Let  $w_\ell$  denote the longest element of the Weyl group. Then  $w_\ell M_{\mathcal{O}} w_\ell^{-1}$  is a standard maximal Levi subgroup. Let  $P_{\mathcal{O}}^a$  be the standard maximal parabolic subgroup of  $H$  such that  $M = w_\ell M_{\mathcal{O}} w_\ell^{-1}$ . (Among exceptional groups,  $P_{\mathcal{O}}^a$  is equal to  $P_{\mathcal{O}}$  except in some cases in  $E_6$ .) We consider the integral (13) in the special case when the parabolic subgroup  $P = MU$  from which we form the Eisenstein series  $E_\tau(h, s)$  is equal to  $P_{\mathcal{O}}^a$ . Thus  $\tau$  is an automorphic representation of  $M(\mathbb{A})$  and we assume that identity (22) holds

$$(34) \quad \dim C + \dim U_{\mathcal{O}} = \dim \pi + \dim E_\tau(\cdot, s)$$

Unfolding the Eisenstein series, we consider the space  $P(F) \backslash H(F) / P_{\mathcal{O}}(F)$ . Let  $w_0$  be longest Weyl element in this space. From the relation  $P = P_{\mathcal{O}}^a$ , we deduce that  $U_{\mathcal{O}, w_0} = U_{\mathcal{O}}$  and that  $Q_{w_0} = M_{\mathcal{O}}$ . Also,  $w_0^{-1} M_{\mathcal{O}} w_0 \cap M_{\mathcal{O}} = M_{\mathcal{O}}$ . Hence, the space  $Q_{w_0} \backslash M_{\mathcal{O}} / C$  contains one element, and we define  $\nu_0 = e$ . From this we deduce that integral (28) is given by

$$\int_{C(F) \backslash C(\mathbb{A})} \varphi_\pi(t) \varphi_\tau(t) dt$$

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We claim that identity (30) holds in this case. Indeed, in this case, we have  $L_{w_0} = C$  and  $V = \{e\}$ , where the last equality follows from the fact that  $U_{\mathcal{O}, w_0} = U_{\mathcal{O}}$ . Recall that  $\dim E_{\tau}(\cdot, s) = \dim \tau + \dim U$ . We have  $\dim U = \dim U_{\mathcal{O}}$ . Plugging this into equation (34) we obtain

$$\dim C + \dim U_{\mathcal{O}} = \dim \pi + \dim \tau + \dim U_{\mathcal{O}}$$

Hence equation (30) holds.

We remark that integrals of this type are almost never preWhittaker.

## 6. Maximal parabolic subgroups of $F_4$

The group  $F_4$  has four standard maximal parabolic subgroups,  $P_1, P_2, P_3$  and  $P_4$ . (See section 2.)

Levi	Isomorphic to...
$M_1$	$GS p_6$
$M_2$	$\{(g_1, g_2) \in GL_2 \times GL_3 \mid \det g_1 \cdot \det g_2 = 1\}$
$M_3$	$\{(g_1, g_2) \in GL_3 \times GL_2 \mid \det g_1 \cdot \det g_2^2 = 1\}$
$M_4$	$GSpin_7$

Before working an example it will be convenient to pin down a specific identification each standard maximal Levi subgroups with the matrix group listed above.

We realize the group  $GS p_6$  as the group generated by  $Sp_6 = \{m \in GL_6 : {}^t m J_6 m = J_6\}$  and the one dimensional torus  $\text{diag}(a, a, a, 1, 1, 1)$ . Here

$$J_6 = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & -1 & & & \\ & -1 & & & & \\ -1 & & & & & \end{pmatrix}$$

We fix specific isomorphisms by mapping

$\boxed{M_1 :}$

$$x_{\alpha_4}(r) \mapsto \begin{pmatrix} 1 & r & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & -r \\ & & & & & 1 \end{pmatrix}, \quad x_{\alpha_3}(r) \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & r & & & \\ & & 1 & & & \\ & & & 1 & -r & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$$x_{\alpha_2}(r) \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & r & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

$$\alpha_1^\vee(t_1)\alpha_2^\vee(t_2)\alpha_3^\vee(t_3)\alpha_4^\vee(t_4) \mapsto \begin{pmatrix} t_4 & & & & \\ & t_4^{-1}t_3 & & & \\ & & t_3^{-1}t_2 & & \\ & & & t_2^{-1}t_3t_1 & \\ & & & & t_3^{-1}t_1t_4 \\ & & & & & t_4^{-1}t_1 \end{pmatrix}$$

**M<sub>2</sub> :**

$$x_{\alpha_1}(r) \mapsto \left( \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, I_3 \right), \quad x_{\alpha_3}(r) \mapsto \left( I_2, \begin{pmatrix} 1 & r \\ & 1 \\ & & 1 \end{pmatrix} \right), \quad x_{\alpha_4}(r) \mapsto \left( I_2, \begin{pmatrix} 1 & & \\ & 1 & r \\ & & 1 \end{pmatrix} \right),$$

$$\alpha_1^\vee(t_1)\alpha_2^\vee(t_2)\alpha_3^\vee(t_3)\alpha_4^\vee(t_4) \mapsto \left( \begin{pmatrix} t_1 & & \\ & t_1^{-1}t_2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} t_2^{-1}t_3 & & \\ & t_3^{-1}t_4 & \\ & & t_4^{-1} \end{pmatrix} \right).$$

**M<sub>3</sub> :**

$$x_{\alpha_1}(r) \mapsto \left( \begin{pmatrix} 1 & r & \\ & 1 & \\ & & 1 \end{pmatrix}, I_2 \right), \quad x_{\alpha_2}(r) \mapsto \left( \begin{pmatrix} 1 & & \\ & 1 & r \\ & & 1 \end{pmatrix}, I_2 \right), \quad x_{\alpha_4}(r) \mapsto \left( I_3, \begin{pmatrix} 1 & r \\ & 1 \end{pmatrix} \right),$$

$$\alpha_1^\vee(t_1)\alpha_2^\vee(t_2)\alpha_3^\vee(t_3)\alpha_4^\vee(t_4) \mapsto \left( \begin{pmatrix} t_1 & & \\ & t_1^{-1}t_2 & \\ & & t_2^{-1}t_3^2 \end{pmatrix}, \begin{pmatrix} t_4t_3^{-1} & & \\ & t_4^{-1} & \\ & & 1 \end{pmatrix} \right).$$

The group  $M_4$  is isomorphic to  $\mathrm{GSpin}_7$ , as mentioned above. This group has an embedding into  $SO_8$ . It also has a projection to  $SO_7$  which restricts to an isomorphism on unipotent elements. We define  $SO_8$  relative to the matrix  $J_8$  which has ones on the diagonal which goes from top right to bottom left and zeros everywhere else. We define  $e_{ij}$  to be the matrix with a 1 at the  $i, j$  and zeros elsewhere, and  $e'_{ij} = e_{ij} - e'_{9-j, 9-i}$ . We fix an isomorphism from  $M_4$  to  $\mathrm{GSpin}_7 \subset SO_8$  as follows:

**M<sub>4</sub> :**

$$x_{\alpha_1}(r) \mapsto I_8 + re'_{12} + re'_{35}, \quad x_{\alpha_2}(r) \mapsto I_8 + re'_{23}, \quad x_{\alpha_3}(r) \mapsto I_8 + re'_{34},$$

$$\alpha_1^\vee(t_1)\alpha_2^\vee(t_2)\alpha_3^\vee(t_3)\alpha_4^\vee(t_4) \mapsto \mathrm{diag}(t_1t_4^{-1}, t_1^{-1}t_2, t_2^{-1}t_1t_3, t_1^{-1}t_3, t_3^{-1}t_1t_4, t_2t_1^{-1}t_3^{-1}t_4, t_1t_2^{-1}t_4, t_1^{-1}t_4^2).$$

When working with unipotent subgroups of  $M_4$ , we often find it more convenient to identify them with their images in  $SO_7$  under a fixed projection. We define  $SO_7$  relative to the matrix  $J_7$  which has ones on the diagonal which goes from top right to bottom left and zeros everywhere else. We fix the projection  $M_4 \rightarrow SO_7$  in such a fashion that

$$x_{\alpha_1}(r_1) \mapsto I_7 + r_1e'_{12}, \quad x_{\alpha_2}(r_2) \mapsto I_7 + r_2e'_{23}, \quad x_{\alpha_3}(r_3) \mapsto I_7 + r_3e_{34} - r_3e_{45} - \frac{r_3^2}{2}e_{35},$$

where now  $e'_{ij} = e_{ij} - e_{8-j, 8-i}$ . This determines the image of  $x_\alpha(r)$  for any positive root  $\alpha$ , and we require that the projection of  $x_{-\alpha}(r)$  is  ${}^tx_\alpha(r)$  for each root  $\alpha$ . This then determines

a unique map  $M_4 \rightarrow SO_7$  which, as we say, is not an isomorphism but gives an isomorphism when restricted to any unipotent subgroup.

## 7. An Example

Let  $H = F_4$ , and consider the unipotent orbit  $\mathcal{O} = \tilde{A}_2$ . The weighted Dynkin diagram for this orbit is

$$\begin{array}{ccccccc} \alpha_1 & & & & \alpha_2 & & \alpha_3 & & \alpha_4 \\ 0 & - & - & - & 0 & == > == & 0 & - & - & - & 2. \end{array}$$

From the table at the end of section 4, it follows that  $C$  is the exceptional group  $G_2$ . We have  $P_{\mathcal{O}} = M_{\mathcal{O}} \cdot U_{\mathcal{O}}$ , with  $M_{\mathcal{O}} \cong \mathrm{GSpin}_7$ . Since this orbit is labeled with zeros and twos only, the integral to consider is integral (13). Write  $L_{\mathcal{O}}^{1,*}$  for the rational representation of  $M_{\mathcal{O}}$  which is dual to the rational representation of  $M_{\mathcal{O}}$  on  $L_{\mathcal{O}}^{(1)}$  given by conjugation. The  $F$ -points of this representation may be identified with the space of characters of  $U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})$ . The action of  $M_{\mathcal{O}}$  on  $L_{\mathcal{O}}^{1,*}$  may be identified with the representation of  $\mathrm{GSpin}_7$  in which  $\mathrm{Spin}_7$  acts by the spin representation and scalars act by scalar multiplication. There is a nondegenerate quadratic form on  $L_{\mathcal{O}}^{1,*}$  which is preserved by the action of  $\mathrm{Spin}_7$ . An element of  $L_{\mathcal{O}}^{1,*}$ , (or its dual, the space of characters) is in general position if its image under this quadratic form is nonzero. Further, the elements of  $L_{\mathcal{O}}^{1,*}(F)$  in general position are a single  $M_{\mathcal{O}}(F)$ -orbit. See [S-K], pp. 114-116, and [I], §4 especially proposition 4.

We choose the character  $\psi_{U_{\mathcal{O}}}$  as follows. For  $u \in U_{\mathcal{O}}$  write  $u = x_{1111}(r_1)x_{0121}(r_2)u'$  as in subsubsection 2.0.1. Then  $\psi_{U_{\mathcal{O}}}(x_{1111}(r_1)x_{0121}(r_2)u') = \psi(r_1 + r_2)$ . It is not hard to check that this character corresponds to a point in general position. The embedding of  $G_2$  into  $\mathrm{GSpin}_7$  as the stabilizer of  $\psi_{U_{\mathcal{O}}}$  is a pretty standard embedding. A maximal unipotent of this stabilizer is generated by all elements of the form

$$\begin{array}{lll} x_{1000}(r)x_{0010}(r), & x_{0100}(r), & x_{1100}(r)x_{0110}(-r), \\ x_{1110}(r)x_{0120}(-r), & x_{1120}(r), & x_{1220}(r), \end{array} \quad (r \in \mathbb{G}_a).$$

For  $1 \leq i \leq 4$ , we let  $w[i]$  be the reflection in the Weyl group attached to the simple root  $\alpha_i$ . We use the same notation for the standard representative for this reflection, which is  $x_{\alpha_i}(1)x_{-\alpha_i}(-1)x_{\alpha_i}(1)$ . Finally,  $w[i_1, i_2, \dots, i_k]$  denotes the product of the corresponding simple reflections in the Weyl group, or of their representatives in  $H(F)$ . Note that the Weyl group of  $G_2$  may be identified with the subgroup of that of  $F_4$  which is generated by the simple reflection  $w[2]$  and the product  $w[1, 3]$ .

Thus, integral (13) is given by

$$I = \int_{G_2(F) \backslash G_2(\mathbb{A})} \int_{U_{\mathcal{O}}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_{\pi}(g) E_{\tau}(ug, s) \psi_{U_{\mathcal{O}}}(u) du dg.$$

The table at the end of section 4 indicates that each of the four parabolic subgroups of  $F_4$  is an option for the parabolic subgroup  $P$  used to form the Eisenstein series. So we have four cases to check. In each case we carry out the unfolding process using the notations of section 5.

7.1.  $\mathbf{P} = \mathbf{P}_1$ . First we determine the space  $P(F) \backslash H(F) / P_{\mathcal{O}}(F)$ . A set of representatives is  $e, w[1, 2, 3, 4]$  and  $w_0 = w[1, 2, 3, 2, 1, 4, 3, 2, 3, 4]$ . Hence we obtain that  $Q_{w_0}$  is the maximal parabolic subgroup of  $\mathrm{GSpin}_7$  whose Levi part is isomorphic to  $GL_1 \times \mathrm{GSpin}_5$ . The group  $U_{\mathcal{O}, w_0}$  is generated by elements  $x_{\alpha}(r)$ , where  $\alpha$  is in the set

$$\{(0001); (0011); (0111); (0121); (0122); (1122); (1222); (1232); (1242); (1342)\}.$$

Next we consider the space  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F) / G_2(F)$ . Consider the action of  $\mathrm{GSpin}_7(F)$  on the projective space consisting of all one-dimensional subspaces of its standard representation. Then  $Q_{w_0}(F)$  is the stabilizer of the line consisting of all highest-weight vectors. Hence, the space  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F)$  may be identified with the orbit of this line. It is not hard to see that this orbit consists of all nonzero vectors “of length 0,” i.e., on which the  $\mathrm{Spin}_7$ -invariant quadratic form vanishes. Now,  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F) / G_2(F)$  may be identified with the set of  $G_2(F)$ -orbits in this  $\mathrm{GSpin}_7(F)$ -orbit. But again one easily checks that  $G_2(F)$  acts transitively on the nonzero vectors of length zero in the standard representation. Thus  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F) / G_2(F)$  has only one element and we may take  $\nu_0 = e$ .

Let  $R$  denote the parabolic subgroup of  $G_2$  which preserves the line of highest weight vectors in the action considered above (i.e., in the standard representation of  $G_2$ ). Thus,  $R$  is the standard maximal parabolic subgroup of  $G_2$  which contains the  $SL_2$  which is generated by  $U_{\pm(0100)}$ . Then in this case  $L_{\nu_0} = R$ . To determine the group  $V$  we first observe that  $U_{\mathcal{O}}^{w_0}$  is the group generated by all  $U_{\alpha}$  such that  $\alpha$  is in the set  $\{(1111); (1121); (1221); (1231); (2342)\}$ . When conjugating by  $w_0$  we obtain the group  $V$  which is the group generated by all  $U_{\alpha}$  such that  $\alpha$  is in the set  $\{(0001); (0011); (0111); (0121); (0122)\}$ . Particular attention must be paid to  $U_{1111}$  because it is not in the kernel of  $\psi_U$ . We have

$$(35) \quad w_0 x_{1111}(r) w_0^{-1} = x_{0001}(r).$$

We identify the Levi part of  $P$  with the group  $GSp_6$  as in section 6, and compute that  $m(v)$  is embedded inside  $GSp_6$  as the group  $Z$  which consists of all matrices in  $GSp_6$  of the form

$$(36) \quad \left\{ z = \begin{pmatrix} 1 & x & y \\ & I_4 & x^* \\ & & 1 \end{pmatrix} : x \in Mat_{1 \times 4}; y \in Mat_{1 \times 1} \right\}.$$



To describe the projection of  $R$  in  $GS p_6$  we first notice that  $w_0 x_{\pm 0100} w_0^{-1} = x_{\pm 0100}$ . Also,  $w_0 1010^\vee(t) w_0 = 1121^\vee(t)$ . Here  $1010^\vee(t) := \alpha_1^\vee(t) \alpha_3^\vee(t)$ , and  $1121^\vee$  is defined similarly. Hence the Levi part of  $R$ , which is isomorphic to the group  $GL_2$  is embedded in  $GS p_6$  as

$$\left\{ \begin{pmatrix} |g|I_2 & & \\ & g & \\ & & I_2 \end{pmatrix} : g \in GL_2 \right\}$$

As for the unipotent radical of  $R$ , which we denote by  $U_R$ , let

$$(37) \quad u = x_{1000}(r_1) x_{0010}(r_1) x_{1100}(r_2) x_{0110}(-r_2) x_{1110}(r_3) x_{0120}(-r_3) x_{1120}(r_4) x_{1220}(r_5)$$

Then conjugating by  $w_0$ , we obtain

$$w_0 u w_0^{-1} = x_{1122}(r_1) x_{0010}(r_1) x_{1222}(r_2) x_{0110}(-r_2) x_{1232}(r_3) x_{0120}(-r_3) x_{1242}(r_4) x_{1342}(r_5).$$

Projecting into  $M$ , we get  $x_{0010}(r_1) x_{0110}(-r_2) x_{0120}(-r_3)$ . Using our identification of  $M$  with  $6 \times 6$  matrices, we get

$$m(w_0 u w_0^{-1}) = \begin{pmatrix} 1 & & & & \\ & 1 & r_1 & r_2 & -r_3 \\ & & 1 & & r_2 \\ & & & 1 & -r_1 \\ & & & & 1 \\ & & & & & 1 \end{pmatrix}$$

Combining all of the above, integral (28) in this case is

$$\int_{GL_2(F) \backslash GL_2(\mathbb{A})} \int_{U_R(F) \backslash U_R(\mathbb{A})} \int_{Z(F) \backslash Z(\mathbb{A})} \varphi_\pi(ut) \varphi_\tau(z m(w_0 u w_0^{-1}) t) \psi_Z(z) dz du dt$$

Here  $\psi_Z$  is defined as follows. For  $z$  as described in (36), we set  $\psi_Z(z) = \psi(x_{1,1})$ . (It follows from (35) that  $\psi_Z(z) = \psi_{U_\mathcal{O}}(w_0^{-1} z w_0)$ .)

We count dimensions. We have  $L_{\nu_0} = R$  and hence  $\dim L_{\nu_0} = 9$ . Also,  $\dim U_{\mathcal{O}}^{w_0} = \dim V = 5$ . Hence  $\dim L_{\nu_0} + \dim V = 14$ . Since  $\pi$  is a generic cuspidal representation of  $G_2$ , its dimension is 6, and as follows from the above table, we assumed that  $\tau$  is attached to the unipotent orbit (42) of  $GS p_6$  whose dimension when divided by two is 8. Hence  $\dim \pi + \dim \tau = 6 + 8 = 14$ . Hence (30) holds.

We state some conclusions. First, it is immediate from the discussion above that integral (28) is of open orbit type. Second, it is not hard to check that the contribution to the integral  $I$  from  $e$  and  $w[1, 2, 3, 4]$  is zero, so conjecture 31 holds in this case.

By choosing  $\tau$  to be a certain Eisenstein series we can further unfold the last integral and obtain an integral with a Whittaker coefficient of  $\pi$ . This shows that the integral is a weakly preWhittaker integral. See definition 5. Since we are not aware of any way to unfold the

above integral, to obtain a Whittaker coefficient of  $\pi$ , and this for any representation  $\tau$ , we conjecture that the integral is not preWhittaker.

7.2. **P = P<sub>2</sub>**. In this case the space  $P(F) \backslash H(F) / P_{\mathcal{O}}(F)$  consists of 5 elements. We can choose as representatives the Weyl elements

$$e; w[2, 3, 4]; w[2, 1, 3, 2, 3, 4]; w[2, 3, 4, 2, 1, 3, 2, 3, 4]; w_0 = w[2, 3, 1, 2, 3, 4, 3, 2, 1, 3, 2, 3, 4].$$

Hence, if we consider  $w_0$ , then  $Q_{w_0}$  is the maximal parabolic subgroup of  $\mathrm{GSpin}_7$  whose Levi part  $M_{w_0}$  is  $GL_2 \times \mathrm{GSpin}_3$ . Also  $U_{\mathcal{O}}^{w_0}$  is generated by the elements  $x_{\alpha}(r)$  such that  $\alpha \in \{(1221); (1231)\}$ . The group  $U_{\mathcal{O}, w_0}$  is the product of the groups  $U_{\alpha}$  for the other 13 roots of  $T$  in  $U_{\mathcal{O}}$ , which are

$$0001, 0011, 0111, 0121, 0122, 1111, 1121, 1122, 1222, 1232, 1242, 1342, 2342$$

Next we consider the space  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F) / G_2(F)$ . This space can be identified with the set of  $Q_{w_0}(F)$ -orbits in the  $\mathrm{GSpin}_7(F)$ -orbit of the character  $\psi_{U_{\mathcal{O}}}$ . Recall that  $\psi_{U_{\mathcal{O}}}$  is identified with a point  $x_0$  in the representation of  $\mathrm{GSpin}_7$  in which  $\mathrm{Spin}_7$  acts by the spin representation and scalars act by scalar multiplication. This is an eight dimensional representation of  $\mathrm{GSpin}_7$ . This representation has a  $\mathrm{Spin}_7$ -invariant bilinear form  $(, )$ . Assume that  $(, )$  is normalized so that  $(x_0, x_0) = 1$ . Then the  $\mathrm{GSpin}_7$  orbit is  $\mathcal{X}_{\square} := \{x : (x, x) \text{ is a square}\}$ . For a suitably chosen basis,  $x_0 = {}^t(0, 0, 0, 1, 1, 0, 0, 0)$  and the image of  $Q_{w_0}$  consists of  $8 \times 8$  matrices of the form

$$\begin{pmatrix} {}^t g_1^{-1} \det g_1 & x & * \\ & g_1 \otimes g_2 & x' \\ & & g_1 \det g_2 \end{pmatrix}, \quad g_1, g_2 \in GL_2, x \in \mathrm{Mat}_{2 \times 4},$$

where  $x'$  is determined by  $x, g_1$  and  $g_2$ , and  $g_1 \otimes g_2$  is given by

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} g_2 \\ g_2 \end{pmatrix}, \quad g_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here and throughout,  ${}^t A$  denotes the transpose of the matrix  $A$  over the diagonal which runs from top right to lower left. Thus  ${}^t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ . We also introduce the notation  $g^* := {}^t g^{-1}$ . Thus  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \frac{1}{ad-bc} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ . From this description it's easy to see that  $\mathcal{X}_{\square}(F)$  is a union of two  $Q_{w_0}(F)$ -orbits: one consisting of those elements of the form  ${}^t(x, y, z)$  with  $x, z \in F^2, y \in F^4$ , such that  $z \neq 0$ , and the other of those such that  $z = 0$ . Clearly, the first of these two orbits is open. One may check that  $\nu_0 = w[2, 1]$  is an element of it. In this case  $L_{\nu_0} = \nu_0^{-1} Q_{w_0} \nu_0 \cap G_2 = R^0$  which is defined as follows. Take the subgroup  $R$  defined as in the case  $P = P_1$ . Then  $R^0 \subset R$  consists of  $M_R \cong GL_2$  and all elements  $u \in U_R$  as in (37)

with  $r_1 = r_2 = 0$ . Thus  $\dim L_{\nu_0} = 7$ . Thus, in this case, integral (28) is given by

$$(38) \quad \int_{GL_2(F) \backslash GL_2(\mathbb{A})} \int_{(F \backslash \mathbb{A})^5} \varphi_\pi(x_{1110}(r_1)x_{0120}(-r_1)x_{1120}(r_2)x_{1220}(r_3)t) \times \\ \varphi_{\tau_1}(t)\varphi_{\tau_2} \begin{pmatrix} 1 & r_1 & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1) dy_i dr_j dt$$

Here we wrote  $\tau = \tau_1 \times \tau_2$  where  $\tau_1$  is an automorphic representation of  $GL_2(\mathbb{A})$ , and  $\tau_2$  is an automorphic representation of  $GL_3(\mathbb{A})$ . (Cf. section 6.) Also, one can check that all other elements give zero contribution to  $I$ . Hence conjecture 31 holds in this case.

We count dimensions. We have  $\dim L_{\nu_0} + \dim V = 7 + 2 = 9$ . According to the table the basic equation implies  $\dim \tau = 3$ , and then formula (30) holds as well.

There are two possibilities for the pair  $(\dim \tau_1, \dim \tau_2)$ , namely  $(1, 2)$  and  $(0, 3)$ . We now treat each of these two possibilities.

7.2.1.  $\dim \tau_1 = 1, \dim \tau_2 = 2$ . If  $\dim \tau_1 = 1$  and  $\dim \tau_2 = 2$ , then  $\tau_1$  is generic and  $\tau_2$  is attached to the orbit (21) of  $GL_3$ . We will show that integral (38) can be further expanded and unfolded to the Whittaker coefficient of  $\pi$  with the only assumption on  $\tau$  that  $\mathcal{O}(\tau_2) = (21)$ . This will prove that integral (28) is preWhittaker in this case. Indeed, we have

$$(39) \quad \int_{(F \backslash \mathbb{A})^2} \varphi_{\tau_2} \begin{pmatrix} 1 & r_1 & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1) dy_1 dy_2 = \sum_{\alpha \in F} \int_{(F \backslash \mathbb{A})^3} \varphi_{\tau_2} \begin{pmatrix} 1 & r_1 + z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1 + \alpha z) dz dy_1 dy_2$$

If  $\alpha \in F^*$ , then we obtain the Whittaker coefficient of  $\tau_2$  as an inner integration. This is zero from the assumption that  $\mathcal{O}(\tau_2) = (21)$ . Hence we are left with the contribution  $\alpha = 0$ . Thus, changing variables, integral (38) is equal to

$$(40) \quad \int_{GL_2(F) \backslash GL_2(\mathbb{A})} \int_{(F \backslash \mathbb{A})^6} \varphi_\pi(x_{1110}(r_1)x_{0120}(-r_1)x_{1120}(r_2)x_{1220}(r_3)t) \times \\ \varphi_{\tau_1}(t)\varphi_{\tau_2} \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1) dz dy_i dr_j dt$$

Next we expand  $\varphi_\pi$  along the group  $U_R/U_R^0$ . We recall that  $U_R$  is the unipotent radical of the maximal parabolic subgroup of  $G_2$  which preserves a line, and  $U_R^0$  is the unipotent radical of the group  $R^0$  defined above. The group  $GL_2(F)$ , as embedded in  $R^0$  acts on this expansion with two orbits. The trivial orbit contributes zero by cuspidality. Thus, the above integral

is equal to

$$(41) \quad \int_{B_0(F) \backslash GL_2(\mathbb{A})} \int_{U_R(F) \backslash U_R(\mathbb{A})} \varphi_\pi(ut) \psi_{U_R}(u) du \varphi_{\tau_1}(t) \int_{(F \backslash \mathbb{A})^3} \varphi_{\tau_2} \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1) dz dy_i dt$$

Here  $\psi_{U_R}$  is defined as follows. If we write  $u$  as a product given in (37), then  $\psi_{U_R}(u) = \psi(r_1)$ . Also, in integral (41) the group  $B_0$  is the subgroup of  $GL_2$  consisting of all matrices of the form  $\begin{pmatrix} a & r \\ & 1 \end{pmatrix}$  where  $a \in F^*$  and  $r \in F$ . This group has the factorization  $B_0 = TN$  where  $T$  is a torus and  $N$  is unipotent. In terms of the group  $F_4$ , the group  $N$  is identified with  $U_{0100}$ . Factoring the measure in integral (41) we obtain

$$(42) \quad \int_{T(F)N(\mathbb{A})(F) \backslash GL_2(\mathbb{A})} \left[ \int_{U_R(F) \backslash U_R(\mathbb{A})} \int_{F \backslash \mathbb{A}} \varphi_\pi(ux_{0100}(r)t) \psi_{U_R}(u) \varphi_{\tau_1}(x_{0100}(r)t) dr du \right] dt \times \\ \int_{(F \backslash \mathbb{A})^3} \varphi_{\tau_2} \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1) dz dy_i$$

Next we plug the Fourier expansion of  $\varphi_\pi$  along  $U_{0100}$  into the expression in brackets. It is equal to

$$\sum_{\alpha \in F} \int_{(F \backslash \mathbb{A})^2} \int_{U_R(F) \backslash U_R(\mathbb{A})} \varphi_\pi(ux_{0100}(r+z)t) \psi_{U_R}(u) \psi(\alpha z) \varphi_{\tau_1}(x_{0100}(r)t) dz dr du$$

It follows from the cuspidality of  $\pi$  that the contribution from  $\alpha = 0$  is zero. Plugging the summation in integral (42), collapsing summation and integration, we obtain the integral

$$\int_{N(\mathbb{A})(F) \backslash GL_2(\mathbb{A})} \int_{(F \backslash \mathbb{A})^3} W_{\varphi_\pi}(t) W_{\varphi_{\tau_1}}(t) \varphi_{\tau_2} \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \psi(y_1) dz dy_i dt$$

Here, for a vector  $\varphi_\sigma$  in a generic representation  $\sigma$  we denote by  $W_{\varphi_\sigma}$  its Whittaker coefficient, with the precise generic character of the maximal unipotent subgroup to be deduced from context. Thus integral (28) is preWhittaker in this case.

7.2.2.  $\dim \tau_1 = 0, \dim \tau_2 = 3$ . (Equivalently,  $\tau_1$  is a character, and  $\tau_2$  is generic.) In this case, we begin in the same fashion, by plugging in the expansion (39). The group  $GL_2$  (once identified with the Levi of the parabolic subgroup  $R \subset C$ ) acts on the characters in this expansion with two orbits. That is,  $GL_2$  normalizes  $U_{1110}$  and if we define  $\psi_a(x_{1110}(r)) = \psi(ar)$ , then  $\psi_a(gx_{1110}(r)g^{-1}) = \psi_{a \det g}(r)$ . It follows that in this case  $I_{w_0, \nu_0}$  can be written as

$I_{w_0, \nu_0}^0 + I_{w_0, \nu_0}^1$ , where

$$I_{w_0, \nu_0}^0 = \int_{GL_2(F)V(\mathbb{A}) \backslash G_2(\mathbb{A})} \int_{(F \backslash \mathbb{A})^3} \varphi_\pi(x_{1110}(r_1)x_{0120}(-r_1)x_{1120}(r_2)x_{1220}(r_3)g)\psi(-r_1) dr \times$$

$$\int_{(F \backslash \mathbb{A})^3} f_\tau \left( \left( I_2, \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \right) g, s \right) \psi(y_1) dz dy_i dg$$

$$I_{w_0, \nu_0}^1 = \int_{SL_2(F)V(\mathbb{A}) \backslash G_2(\mathbb{A})} \int_{(F \backslash \mathbb{A})^3} \varphi_\pi(x_{1110}(r_1)x_{0120}(-r_1)x_{1120}(r_2)x_{1220}(r_3)g)\psi(-r_1) dr \times$$

$$\int_{(F \backslash \mathbb{A})^3} f_\tau \left( \left( I_2, \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \right) g, s \right) \psi(z + y_1) dz dy_i dg.$$

Here  $V = U_{1110}U_{0120}U_{1120}U_{1220}$ .

**Lemma 43.** The integral  $I_{w_0, \nu_0}^0$  is zero for all cusp forms  $\varphi_\pi$  and all  $f_\tau \in \text{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})} \tau \delta_P^s$ , where  $s \in \mathbb{C}$  and  $\tau$  is attached to the orbit  $(1^2|3)$  of  $GL_2 \times GL_3$ .

*Proof.* The proof relies on the fact that  $\varphi_\pi$  is cuspidal, while  $f_\tau$  is invariant by  $U_{1000}$ . First expand

$$\varphi_\pi^V := \int_{V(F) \backslash V(\mathbb{A})} \varphi_\pi(vg) dg$$

along  $U_{1000}U_{1100}(F) \backslash U_{1000}U_{1100}(\mathbb{A})$ . The constant term in this Fourier expansion is a constant term of  $\varphi_\pi$ , and hence zero. The remaining terms are permuted transitively by  $GL_2$  (embedded as a subgroup of  $R$ ). As a representative we choose  $x_{1000}(r_{1000})x_{1100}(r_{1100}) \mapsto \psi(r_{1000})$ , and the stabilizer in  $GL_2$  is the product of  $U_{0100}$  and a one-dimensional torus. Thus

$$I_{w_0, \nu_0}^0 = \int_{T_1(F)U_{0100}V(\mathbb{A}) \backslash G_2(\mathbb{A})} \int \varphi_\pi^{(V', \psi'_V)}(g) \int_{(F \backslash \mathbb{A})^3} f_\tau \left( \left( I_2, \begin{pmatrix} 1 & z & y_2 \\ & 1 & y_1 \\ & & 1 \end{pmatrix} \right) g, s \right) \psi(y_1) dz dy_i dg,$$

where  $V' = U_{1000}U_{1100}V$ , and  $\psi'_V(v') = \psi(v'_{1000})$ . Factor the integration over  $U_{0100}(F) \backslash U_{0100}(\mathbb{A})$ . Since  $w_0\nu_0 \cdot 0100 = 1000$ , it follows that  $h \mapsto f_\tau(w_0\nu_0h, s)$  is invariant by  $U_{0100}(\mathbb{A})$ . And so the function  $\varphi_\pi \mapsto I_{w_0, \nu_0}^0$  factors through the constant term attached to the standard maximal parabolic subgroup of  $C$  whose unipotent radical contains  $U_{0100}$ . This completes the proof.  $\square$

**7.3.  $\mathbf{P} = \mathbf{P}_3$ .** In this case the space  $P(F) \backslash H(F)/P_{\mathcal{O}}(F)$  consists of 7 elements. We can choose as representatives the Weyl elements

$$e; w[3, 4]; w[3, 2, 3, 4]; w[3, 2, 1, 3, 2, 3, 4]; w[3, 2, 1, 4, 3, 2, 3, 4];$$

$$w[3, 2, 3, 4, 3, 2, 1, 3, 2, 3, 4]; w_0 = w[3, 2, 1, 3, 2, 3, 4, 3, 2, 1, 3, 2, 3, 4].$$

The group  $Q_{w_0}$  is the maximal parabolic subgroup of  $\mathrm{GSpin}_7$  whose Levi part is  $GL_3 \times GL_1$ . A similar argument shows that the space  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F) / G_2(F)$  is finite and we may choose  $\nu_0 = w[3, 2, 1]$ . Conjugating by  $w_0\nu_0$  we obtain that  $L_{\nu_0} = SL_3$  embedded in  $G_2$  as the group generated by the groups  $U_\alpha$  attached to the long roots. Hence, integral (28) is given by

$$\int_{SL_3(F) \backslash SL_3(\mathbb{A})} \varphi_\pi(t) \varphi_{\tau_1}(t) dt \int_{F \backslash \mathbb{A}} \varphi_{\tau_2} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \psi(x) dx.$$

Here  $\tau = \tau_1 \times \tau_2$  where  $\tau_1$  is an automorphic representation of  $GL_3$  and  $\tau_2$  of  $GL_2$ . As in the two previous cases conjecture 31 holds in this case, and this is an open orbit type of integral.

**7.4.  $\mathbf{P} = \mathbf{P}_4$ .** This case is an instance of the special case discussed in subsection 5.2. In this case the space  $P(F) \backslash H(F) / P_{\mathcal{O}}(F)$  consists of the following 5 elements,

$$e; w[4]; w[4, 3, 2, 3, 4]; w[4, 3, 2, 1, 3, 2, 3, 4]; w_0 = w[4, 3, 2, 1, 3, 2, 3, 4, 3, 2, 1, 3, 2, 3, 4]$$

Thus  $Q_{w_0} = \mathrm{GSpin}_7$ , and hence  $Q_{w_0}(F) \backslash \mathrm{GSpin}_7(F) / G_2(F)$  consists of one element. Thus  $\nu_0 = e$ , and integral (28) is

$$\int_{G_2(F) \backslash G_2(\mathbb{A})} \varphi_\pi(t) \varphi_\tau(t) dt$$

As before conjecture 31 holds in this case, and this is an open orbit type of integral.

## 8. Dealing with theta functions

**8.1. A lemma on Heisenberg groups.** We often use the fact that some unipotent subgroup of  $F_4$  is isomorphic to a Heisenberg group. The point of this section is to record a simple lemma regarding what choices one has to make in order to pin down a specific isomorphism.

By a Heisenberg group, we mean a two-step nilpotent, unipotent linear algebraic group  $U$  such that the center  $Z$  and the quotient  $U/Z$  are vector groups, and the skew-symmetric form  $U/Z \times U/Z \rightarrow Z$  defined by the commutator is nondegenerate. The standard Heisenberg group  $\mathcal{H}_{2n+1}$  in  $2n+1$  variables is defined by equipping  $\mathbb{G}_a^n \times \mathbb{G}_a^n \times \mathbb{G}_a$  with the operation

$$(x_1|y_1|z_1) \cdot (x_2|y_2|z_2) = \left( x_1 + x_2 \mid y_1 + y_2 \mid z_1 + z_2 + \frac{1}{2}(x_1 \cdot {}_t y_2 - y_1 \cdot {}_t x_2) \right).$$

(For  $(x|y|z) \in \mathcal{H}_{2n+1}$ , we think of  $x$  and  $y$  as row vectors.) We normally write an element of  $\mathcal{H}_{2n+1}$  as  $(x|y|z)$ , with  $x, y \in \mathbb{G}_a^n$  and  $z \in \mathbb{G}_a$ . We shall also use the notation  $(x|y|z)_{2n+1}$  when it is desirable to reflect the size of the group in the notation.

**Lemma 44.** If  $U$  is any Heisenberg group, then  $U$  is isomorphic to the standard Heisenberg group of the same dimension, and an  $F$ -isomorphism is uniquely determined by an ordered collection  $(x_1, \dots, x_n, y_1, \dots, y_n)$  of injective  $F$ -morphisms  $\mathbb{G}_a \rightarrow U$  satisfying

$$(x_i(r), x_j(s)) = 1_U \quad \forall i, j, \quad (x_i(r), y_j(s)) = 1_U \quad \forall i, j \text{ s.t. } i + j \neq n + 1,$$

$$(y_i(r), y_j(s)) = 1_U \quad \forall i, j, \quad (x_i(r), y_{n+1-i}(s)) = (x_j(r), y_{n+1-j}(s)) \neq 1_U \quad \forall i, j,$$

for all  $r, s \in \mathbb{G}_a$ . (Here,  $(\ , \ )$  denotes the commutator and  $1_U$  denotes the identity element of the group  $U$ .)

The proof is straightforward, and it is omitted. We note that an isomorphism  $U \rightarrow \mathcal{H}_{2n+1}$  also determines an isomorphism of the group of all automorphisms of  $U$  which fix  $Z$  pointwise with  $Sp_{2n}$ . Suppose that a Fourier coefficient is defined as in section 3, by exploiting the Heisenberg group structure of  $U$  to define theta functions. Then the group  $C$  acts on  $U$  by automorphisms which fix  $Z$  pointwise, and hence a choice of isomorphism  $U \rightarrow \mathcal{H}_{2n+1}$  also determines a homomorphism  $C \rightarrow Sp_{2n}$ . One may make this description more explicit in terms of the semidirect product  $\mathcal{H}_{2n+1} \rtimes Sp_{2n}$ : given an isomorphism  $l : U \rightarrow \mathcal{H}_{2n+1}$ , there is a unique homomorphism  $\rho : C \rightarrow Sp_{2n}$  such that  $l(gug^{-1}) = \rho(g)l(u)\rho(g^{-1})$  for all  $g \in C$  and  $u \in U$ .

Write  $\Phi(U/Z, T)$  for the set of roots of  $T$  in the quotient  $U/Z$ , i.e., for  $\Phi(U) \setminus \Phi(Z)$ . The simplest way to choose a family of morphisms in lemma 44 is to

- pair up the roots of  $T$  in  $U/Z$ : for each  $\alpha$  in  $\Phi(U/Z, T)$  there is a unique “partner,” such that the sum the unique root of  $T$  in  $Z$ .
- order the roots in such a way that for each  $i$ , the root at position  $2n + 1 - i$  is the partner (as above) of the root at position  $i$ ,
- for  $i = 1$  to  $n$ , let  $x_i = x_{\beta_i}$ , where  $\beta_i$  denotes the root at position  $i$  in the ordering just fixed,
- for  $i = 1$  to  $n$ , let  $y_i(r) = x_{\beta_{2n+1-i}}(r/N(\beta_i, \beta_{2n+1-i}))$ .

Here,  $N(\beta_i, \beta_{2n+1-i})$  denotes the structure constant, defined as in [G-S]. This will be our “standard” method of determining an isomorphism  $U \rightarrow \mathcal{H}_{2n+1}$  based on listing the roots of  $T$  in  $U/Z$  in a specific order. Note that the roots of  $T$  in  $U/Z$  come equipped with a partial ordering associated to the base of simple roots for  $\Phi(G, T)$ . That is,  $\alpha < \beta$  if  $\beta - \alpha$  is a positive root. We shall normally order the elements of  $U/Z$  in a fashion which is compatible with this partial ordering.

As an example, consider the unipotent orbit  $\mathbf{A}_1$  in the exceptional group  $F_4$ . It’s diagram is

$$\begin{array}{ccccccc} \alpha_1 & & & & \alpha_2 & & \alpha_3 & & \alpha_4 \\ 1 & - & - & - & 0 & == > == & 0 & - & - & - & 0. \end{array}$$

The corresponding parabolic subgroup  $P$  is  $P_1$ . The unipotent radical  $U$  of this parabolic subgroup is isomorphic to  $\mathcal{H}_{15}$ . Indeed, it has a one dimensional center, which is the root subgroup  $U_{2342}$ . The other 14 roots of  $T$  in  $U$  are

$$1000, 1100, 1110, 1120, 1111, 1220, 1121, 1221, 1122, 1231, 1222, 1232, 1242, 1342.$$

One easily checks that for each root  $\alpha$  which appears on this list, the “partner”  $2342 - \alpha$  is also present. For example, the partner of 1120 is 1222. Also, the roots are ordered in such a way that each root and its partner are in symmetric positions. For example, 1120 is the fourth root, and 1222 is fourth-last. In order to fix an isomorphism  $l : U \rightarrow \mathcal{H}_{15}$ , we require that  $l(x_{2342}(z)) = (0|0|z)$  and

$$l(x_{1000}(r_1)x_{1100}(r_2)x_{1110}(r_3)x_{1120}(r_4)x_{1111}(r_5)x_{1220}(r_6)x_{1121}(r_7)) = (r_1, r_2, r_3, r_4, r_5, r_6, r_7|0|0).$$

The relevant structure constants from [G-S] are

$$\begin{aligned} N(1000, 1342) &= -1, & N(1100, 1242) &= 1, & N(1110, 1232) &= -2, & N(1120, 1222) &= 1, \\ N(1111, 1231) &= 2, & N(1220, 1122) &= -1, & N(1121, 1221) &= -2. \end{aligned}$$

This means that

$$(x_{1000}(r), x_{1342}(s)) = x_{2342}(-rs), \quad (x_{1121}(r), x_{1221}(s)) = x_{2342}(-2rs), \quad \text{etc.}$$

It follows that the preimage of  $(0|y_1, y_2, y_3, y_4, y_5, y_6, y_7|0)$  under  $l$  must be

$$x_{1221}\left(-\frac{y_1}{2}\right)x_{1122}(-y_2)x_{1231}\left(\frac{y_3}{2}\right)x_{1222}(y_4)x_{1232}\left(-\frac{y_5}{2}\right)x_{1242}(y_6)x_{1342}(-y_7).$$

Also in this case there are six other orderings of the roots which are compatible with the partial ordering inherited from our choice of a base of simple roots:

$$\begin{aligned} &1000, 1100, 1110, 1120, 1220, 1111, 1121, \dots \\ &1000, 1100, 1110, 1120, 1111, 1121, 1122, \dots \\ &1000, 1100, 1110, 1120, 1111, 1121, 1220, \dots \\ &1000, 1100, 1110, 1111, 1120, 1220, 1121, \dots \\ &1000, 1100, 1110, 1111, 1120, 1121, 1122, \dots \\ &1000, 1100, 1110, 1111, 1120, 1121, 1220, \dots \end{aligned}$$

Where “...” indicates that the remaining seven roots are ordered by symmetry.

In practice, it makes sense to vary the choice of ordering (i.e., the choice of isomorphism  $U_{\mathcal{O}} \rightarrow \mathcal{H}_{2n+1}$ ) based on the situation.

Returning to the general case, suppose that if  $l_1, l_2 : U \rightarrow \mathcal{H}_{2n+1}$  are two isomorphisms, defined over  $F$ , such that the restrictions to the center of  $U$  agree. Define  $\rho_1, \rho_2 : C \rightarrow Sp_{2n}$  as before, by requiring that  $\rho_i(g)l_i(u)\rho_i(g^{-1}) = l_i(gug^{-1})$  for  $i = 1, 2$ . Then there is



an element  $\sigma \in Sp_{2n}(F)$  such that  $l_2(u)\rho_2(g) = \sigma l_1(u)\rho_1(g)\sigma^{-1}$ . Hence  $\theta_\phi^\psi(l_1(u)\rho_1(g)) = \theta_{\omega_\psi(\sigma)\phi}^\psi(l_2(u)\rho_2(g))$  for all  $\phi \in S(\mathbb{A}^n)$ . Thus, one may vary the choice of isomorphism  $U \rightarrow \mathcal{H}_{2n+1}$  fairly freely.

We shall refer to an algebraic group  $U$  as a **generalized Heisenberg group** if

- (1) it is two step nilpotent, with center  $Z$  equal to its commutator subgroup  $(U, U)$ ,
- (2) both  $Z$  and  $U/Z$  are “vector groups,”
- (3) there exists a linear form  $\ell : Z \rightarrow \mathbb{G}_a$  such that the skew symmetric bilinear form  $U/Z \rightarrow \mathbb{G}_a$  induced by composing  $\ell$  with the commutator  $U \rightarrow Z$  is nondegenerate.

Note that the existence of  $\ell$  as in the last condition clearly requires  $\dim U/Z$  to be even, and that any choice of  $\ell$  induces a projection from  $U$  onto  $U/\ker \ell$ , which is a Heisenberg group.

**Lemma 45.** Let  $\mathcal{O}$  be an orbit and define  $V_{\mathcal{O}}^{(i)}$  as in section 3. Suppose that  $\ell : V_{\mathcal{O}}^{(2)}/V_{\mathcal{O}}^{(3)} \rightarrow \mathbb{G}_a$  is in general position. Then  $(x, y) \mapsto \ell(xyx^{-1}y^{-1})$  is a nondegenerate skew-symmetric form on  $V_{\mathcal{O}}^{(1)}/V_{\mathcal{O}}^{(2)}$ .

*Proof.* To explain this phenomenon, one needs to review a bit more of the theory underpinning the construction of Fourier coefficients given in section 3. Let  $\mathcal{O}$  be a unipotent conjugacy class in  $H$ . Then,  $\mathcal{O}$  can be identified via the exponential map with an orbit for the adjoint action of  $H$  on its Lie algebra  $\mathfrak{h}$ . Such an orbit can be identified, via the Jacobson-Morozov theorem, with an orbit of Lie algebra homomorphisms  $\mathfrak{sl}_2 \rightarrow \mathfrak{h}$ . Let  $L : \mathfrak{sl}_2 \rightarrow \mathfrak{h}$  be a homomorphism in the orbit attached to  $\mathcal{O}$ , with the property that  $h_L := L\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$  lies in the Lie algebra  $\mathfrak{t}$  of our fixed maximal torus  $T$ . Then the weight of a simple root  $\alpha_i$  in the weighted Dynkin diagram of  $\mathcal{O}$  is simply the eigenvalue for  $h_L$  on the one-dimensional subalgebra  $\mathfrak{h}_{\alpha_i}$ . Furthermore,  $L$  can be chosen so that  $\ell(\exp X) = \beta(X, f_L)$ , for all  $X \in \mathfrak{v}_{\mathcal{O}}^{(2)}$  (the Lie algebra of  $V_{\mathcal{O}}^{(2)}$ ), where  $f_L := L\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ . Write  $\mathfrak{h}_i$  for the  $i$ -eigenspace of  $h_L$  in  $\mathfrak{h}$ . By viewing  $\mathfrak{h}$  as a direct sum of  $L(\mathfrak{sl}_2)$ -modules, we see that  $X \mapsto [f_L, X]$  is an isomorphism  $\mathfrak{h}_1 \rightarrow \mathfrak{h}_{-1}$ . Now, given  $X \in \mathfrak{h}_1$  there exists  $Y \in \mathfrak{h}_{-1}$  such that  $\beta(X, Y) \neq 0$ . But then for  $X' \in \mathfrak{h}_1$  with  $Y = [f_L, X']$ , we have  $\beta(X, [f_L, X']) = \beta([X, X'], f_L) \neq 0$ . Nondegeneracy of the form  $(x, y) \mapsto \ell(xyx^{-1}y^{-1})$  follows easily.  $\square$

The procedure outlined above for fixing a specific isomorphism from a Heisenberg group to the standard Heisenberg group of the appropriate dimension may be used on generalized Heisenberg groups to extend  $\ell$  to a projection  $l : U \rightarrow \mathcal{H}_{2n+1}$  for suitable  $n$ , if the function  $\ell : Z \rightarrow \mathbb{G}_a$  is supported on a single root subgroup  $U_\alpha$ .

As an example consider the unipotent subgroup  $U = U_4$  of  $F_4$ . It is two step nilpotent and 15-dimensional. The center,  $Z$ , is seven-dimensional, and we have

$$\Phi(Z, T) = \{0122, 1122, 1222, 1232, 1242, 1342, 2342\}.$$

The group  $U/Z$  is eight-dimensional, and we have

$$\Phi(U/Z, T) = \{0001, 0011, 0111, 1111, 0121, 1121, 1221, 1231\}$$

Observe that for each  $\alpha \in \Phi(U/Z, T)$ , the root  $1232 - \alpha$  is also in  $\Phi(U/Z, T)$ . Hence we can define a surjective homomorphism  $l : U \rightarrow \mathcal{H}_9$  such that

$$l(x_{0122}(r_1)x_{1122}(r_2)x_{1222}(r_3)x_{1232}(r_4)x_{1242}(r_5)x_{1342}(r_6)x_{2342}(r_7)) = (0|0|r_4).$$

As in the case of an ordinary Heisenberg group, there are many such isomorphisms, and to pin down a specific one, we may use a choice of ordering on the roots and the homomorphisms  $x_\alpha$ , as well as the associated structure constants. For example we may order  $\Phi(U/Z, T)$  as

$$0001, 0011, 0111, 1111, 0121, 1121, 1221, 1231.$$

Note that for each root  $\alpha$ , the “partner” root  $1232 - \alpha$  is in a symmetric position. The relevant structure constants are

$$N(0001, 1231) = -1, \quad N(0011, 1221) = 1, \quad N(0111, 1121) = 1, \quad N(1111, 0121) = -1.$$

So, we may define  $l : U \rightarrow \mathcal{H}_9$  such that

$$l(x_{0001}(r_1)x_{0011}(r_2)x_{0111}(r_3)x_{1111}(r_4)) = (r_1, r_2, r_3, r_4|0|0),$$

$$l(x_{0121}(r_1)x_{1121}(r_2)x_{1221}(r_3)x_{1231}(r_4)) = (0|-r_1, r_2, r_3, -r_4|0).$$

In this case, there is only one other possible ordering, which is

$$0001, 0011, 0111, 0121, 1111, 1121, 1221, 1231.$$

Note that the above construction does not work if 1232 is replaced by one of the other roots in  $\Phi(Z, T)$ : for each other root  $\beta \in \Phi(Z, T)$  there exists  $\alpha \in \Phi(U/Z, T)$  such that  $\beta - \alpha$  is not a root. On the other hand, one can check that

$$l(x_{0122}(z_1)x_{1122}(z_2)x_{1222}(z_3)x_{1232}(z_4)x_{1242}(z_5)x_{1342}(z_6)x_{2342}(z_7)) = (0|0|z_1 + z_7)$$

$$l(x_{0001}(r_1)x_{0011}(r_2)x_{1111}(r_3)x_{1121}(r_4)) = (r_1, r_2, r_3, r_4|0|0),$$

$$l(x_{1221}(s_1)x_{1231}(s_2)x_{0111}(s_3)x_{0121}(s_4)) = (0|-2s_1, 2s_2, -2s_3, -2s_4|0),$$

gives a well defined homomorphism  $l : U \rightarrow \mathcal{H}_9$ . (Note that

$$x_{0122}(z_1)x_{1122}(z_2)x_{1222}(z_3)x_{1232}(z_4)x_{1242}(z_5)x_{1342}(z_6)x_{2342}(z_7) \mapsto z_4$$

and

$$x_{0122}(z_1)x_{1122}(z_2)x_{1222}(z_3)x_{1232}(z_4)x_{1242}(z_5)x_{1342}(z_6)x_{2342}(z_7) \mapsto z_1 + z_7$$

are in general position for the action of  $M_4 \cong \text{GSpin}_7$  on the space of linear maps  $Z \rightarrow \mathbb{G}_a$ , but

$$x_{0122}(z_1)x_{1122}(z_2)x_{1222}(z_3)x_{1232}(z_4)x_{1242}(z_5)x_{1342}(z_6)x_{2342}(z_7) \mapsto z_i$$

is not for  $i \neq 4$ .)

**8.2. Explicit formulae for the Weil representation.** In this short subsection, we record for convenience some explicit formulae for the action of the Weil representation of  $\mathcal{H}_{2n+1}(\mathbb{A}) \rtimes \widetilde{Sp}_{2n}(\mathbb{A})$  which we shall use repeatedly when dealing with cases involving theta functions. A reference is [P]. We write  $\text{Mat}_{n \times n}^0$  for the space of  $n \times n$  matrices  $X$  such that  ${}_tX = X$ .

For  $x, y \in \mathbb{A}^n$  (realized as row vectors),  $z \in \mathbb{A}$ ,  $g \in GL_n(\mathbb{A})$ ,  $\varepsilon = \pm 1$ , and  $X \in \text{Mat}_{n \times n}^0(\mathbb{A})$  :

$$(46) \quad \omega_\psi(0|0|z)\phi = \psi(z)\phi \quad \omega_\psi(x|0|0)\phi(\xi) = \phi(\xi + x) \quad \omega_\psi(0|y|0)\phi(\xi) = \psi(y {}_t\xi)\phi(\xi)$$

$$(47) \quad \omega_\psi\left(\begin{pmatrix} g & \\ & g^* \end{pmatrix}, \varepsilon\right)\phi(x) = \varepsilon \gamma_{\det g} \phi(\xi g) \quad \omega_\psi\begin{pmatrix} I & X \\ & I \end{pmatrix}\phi(\xi) = \psi\left(\frac{1}{2}\xi X {}_t\xi\right)\phi(\xi),$$

where  $\gamma_a$  is given in terms of the function  $\omega$  defined on p. 377 of [?] by the formula  $\gamma_a = \omega(1)\omega(-1/a)$ .

**Lemma 48.** If  $k < n$  are integers, embed  $Sp_{2k} \hookrightarrow Sp_{2n}$  and  $\mathcal{H}_{2k+1} \hookrightarrow \mathcal{H}_{2n+1}$  by

$$g \mapsto \begin{pmatrix} I_{n-k} & & \\ & g & \\ & & I_{n-k} \end{pmatrix}, \quad (x|y|z)_{2k+1} \mapsto (0, x|y, 0|z)_{2n+1}.$$

This induces an embedding  $\iota : \mathcal{H}_{2k+1} \rtimes \widetilde{Sp}_{2k}(\mathbb{A}) \rightarrow \mathcal{H}_{2n+1} \rtimes \widetilde{Sp}_{2n}(\mathbb{A})$  such that  $\omega_\psi^n \circ \iota = \omega_\psi^k$ . Here  $\omega_\psi^n$  and  $\omega_\psi^k$  denote the Weil representations of  $\mathcal{H}_{2n+1} \rtimes \widetilde{Sp}_{2n}(\mathbb{A})$  and  $\mathcal{H}_{2k+1} \rtimes \widetilde{Sp}_{2k}(\mathbb{A})$ , respectively. If  $\phi$  is an element of the Schwartz space  $S(\mathbb{A}^n)$ ,  $u \in \mathcal{H}_{2n+1}(\mathbb{A})$ , and  $\tilde{g} \in \widetilde{Sp}_{2n}(\mathbb{A})$ , define

$$(49) \quad \tilde{\theta}_{\phi|_k}^\psi(u\tilde{g}) := \sum_{\xi \in F^k} \omega_\psi(u\tilde{g})\phi(0, \xi).$$

Then

(i):

$$\int_{(F \setminus \mathbb{A})^{n-k}} \tilde{\theta}_\phi^\psi((0|0, y|0)u\tilde{g})dy = \theta_{\phi|_k}^\psi(u\tilde{g}) \quad (\forall u \in \mathcal{H}_{2n+1}(\mathbb{A}), \tilde{g} \in \widetilde{Sp}_{2n}(\mathbb{A})).$$

(ii): For  $u \in \mathcal{H}_{2k+1}(\mathbb{A})$ ,  $\tilde{g} \in \widetilde{Sp}_{2k}(\mathbb{A})$ , we have

$$\tilde{\theta}_{\phi|_k}^\psi(\iota(u\tilde{g})) = \tilde{\theta}_{\phi_1}^{\psi, 2k}(u\tilde{g}),$$

where  $\tilde{\theta}_{\phi_1}^{\psi, 2k}$  is the theta function on  $\mathcal{H}_{2k+1} \rtimes \widetilde{Sp}_{2k}(\mathbb{A})$  defined using the function  $\phi_1(x) := \phi(0, x)$ , which lies in  $S(\mathbb{A}^k)$ . Here  $x \in \mathbb{A}^k$  and 0 denotes the zero element of  $\mathbb{A}^{n-k}$ .

(iii): The function  $\tilde{\theta}_{\phi|_k}^\psi$  satisfies

$$(50) \quad \tilde{\theta}_{\phi|_k}^\psi \left[ \begin{pmatrix} g_1 & & \\ & I_{2k} & \\ & & {}_t g_1^{-1} \end{pmatrix} \begin{pmatrix} I_{n-k} & X & Y \\ & I_{2k} & X' \\ & & I_{n-k} \end{pmatrix} u\tilde{g} \right] = \gamma_{\det g_1} \tilde{\theta}_{\phi|_k}^\psi(u\tilde{g}),$$

for all  $g_1 \in GL_{n-k}(\mathbb{A})$ ,  $u \in \mathcal{H}_{2n+1}$ ,  $\tilde{g} \in \widetilde{Sp}_{2n}(\mathbb{A})$ , and for all  $X \in \text{Mat}_{(n-k) \times (2k)}(\mathbb{A})$ ,  $X' \in \text{Mat}_{(2k) \times (n-k)}(\mathbb{A})$ ,  $Y \in \text{Mat}_{(n-k) \times (n-k)}(\mathbb{A})$ , such that the matrix above is in  $Sp_{2n}$ .

*Proof.* This follows from the formulas for the Weil representation given above.  $\square$

**8.3. Unfolding in the presence of theta functions.** In this subsection, we give a discussion similar to section 5.1, for an integral of the type defined in equation (14). As in subsection 5.1, we express the integral (14) as a double sum indexed first by  $w \in P \backslash H / P_{\mathcal{O}}$ , and then by  $\nu \in Q_w \backslash M / C$ , with  $Q_w = M \cap w^{-1} P w$ . For  $\nu \in M$  we define  $L_\nu = H \cap (w\nu)^{-1} P (w\nu)$  and  $V = Q \cap w U w^{-1}$ . Then in this case

$$\begin{aligned} I_{w,\nu} &= \int_{L_\nu(F) \backslash C(\mathbb{A})} \int_{U_{\mathcal{O}}^{w\nu}(F) \backslash U_{\mathcal{O}}(\mathbb{A})} \varphi_\pi(g) f_\tau(w\nu u g, s) \tilde{\theta}_\phi^\psi(l(u) \rho(g)) \psi_{U_{\mathcal{O}}}(l'(u)) du dg \\ &= \int_{L_\nu(F) \backslash C(\mathbb{A})} \varphi_\pi(g) \int_{U_{\mathcal{O}}^{w\nu}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} \int_{U_{\mathcal{O}}^{w\nu}(F) \backslash U_{\mathcal{O}}^{w\nu}(\mathbb{A})} f_\tau(w\nu u_1 u g, s) \tilde{\theta}_\phi^\psi(l(u_1 u) \rho(g)) \psi_{U_{\mathcal{O}}}(l'(u_1 u)) du_1 du dg. \end{aligned}$$

Here  $l$  and  $l'$  are the projections of  $U_{\mathcal{O}}/V_{\mathcal{O}}^{(3)}$  onto its Heisenberg and abelian factors as in section 4. The definition of “open orbit type” for an integral of type (14), of course reflects the dimension of the theta representation.

**Definition 51.** An integral of the type (14) is said to be of **open orbit type** if there is a unique  $w_0 \in P \backslash H / P_{\mathcal{O}}$  and a unique  $\nu_0 \in Q_{w_0} \backslash M / C$  such that

$$\dim L_{\nu_0} + \dim U_{\mathcal{O}}^{w_0 \nu_0} = \dim \pi + \dim \tau + \dim \Theta^\psi.$$

It is weakly open orbit type if the number  $\nu_0$  satisfying this identity is greater than one, but the integral  $I_{w_0, \nu_0}$  vanishes for all  $\nu_0$  but one. It is preWhittaker if

- (1) it is weakly open orbit type,
- (2)  $I_{w,\nu}$  vanishes for all pairs  $(w, \nu) \neq (w_0, \nu_0)$ ,
- (3)  $I_{w_0, \nu_0}$  is equal to a Whittaker integral, with no conditions on  $\tau$  beyond those listed in the table at the end of section 4.

It is weakly preWhittaker if this the first two conditions hold, and  $I_{w_0, \nu_0}$  is equal to a Whittaker integral, under some additional condition on  $\tau$  (e.g., that it is an Eisenstein series).

In this case, we shall refer to the integral  $I_{w_0, \nu_0}$  as the “contribution from the open orbit,” and, introduce a corresponding “inner period,” analogous to (28), and given by

$$(52) \quad \int_{L_\nu(F) \backslash L_\nu(\mathbb{A})} \varphi_\pi(h) \int_{U_{\mathcal{O}}^{w\nu}(F) \backslash U_{\mathcal{O}}^{w\nu}(\mathbb{A})} \varphi_\tau(m(w\nu u h (w\nu)^{-1})) \tilde{\theta}_\phi^\psi(l(u)\rho(h)) \psi_{U_{\mathcal{O}}}(l'(u)) du dh.$$

The integral  $I_{w, \nu}$  can be simplified in some cases using lemma 48.

It should be noted that for integrals of this type, it is sometimes necessary to take  $\pi$  to be a genuine representation on a covering group of  $C(\mathbb{A})$ . In this case, uniqueness of local Whittaker models, and as a consequence Whittaker integrals need not be Eulerian. Thus, the question of whether a given integral is preWhittaker may not be directly relevant to the question of whether it is Eulerian. Nevertheless, particularly in view of [BBCFH], [BBF1], [BBFH], [], it seems that preWhittaker integrals which involve a cusp form on a covering group are certainly not without interest.

## 9. An example with a theta function

Take  $\mathcal{O} = \mathbf{A}_1$ . The corresponding weighted Dynkin diagram is

$$\begin{array}{ccccccc} \alpha_1 & & & \alpha_2 & & \alpha_3 & \alpha_4 \\ 1 & - & - & - & 0 & == > == & 0 & - & - & - & 0. \end{array}$$

Hence  $P_{\mathcal{O}} = P_1$ . The unipotent radical of  $P_1$  is a Heisenberg group in 15 variables. The center is  $U_{2342}$ . The Levi,  $M_{\mathcal{O}}$  is isomorphic to  $GS p_6$ .

Define a character of  $U_{\mathcal{O}}^{(2)}$  by  $\psi_{U_{\mathcal{O}}}(u) = \psi(u_{2342})$ . The stabilizer  $C$  of this character in  $M_{\mathcal{O}}$  is the derived group of  $M_{\mathcal{O}}$ , isomorphic to  $Sp_6$ . We identify it with  $Sp_6$  using the isomorphism  $M_{\mathcal{O}} \rightarrow GS p_6$  given in section 6.

Write  $\Theta_7^\psi$  for the theta representation of  $\mathcal{H}_{15}(\mathbb{A}) \rtimes \widetilde{Sp}_{14}(\mathbb{A})$ , and let  $l : U_{\mathcal{O}} \rightarrow \mathcal{H}_{15}$  be an isomorphism such that  $l(x_{2342}(z)) = (0|0|z)$ . It will be convenient to choose different isomorphisms  $l$  for different cases below. The isomorphism  $l$  also determines a homomorphism  $C \rightarrow Sp_{14}$ . Since the action of  $C$  on  $U_{\mathcal{O}}/U_{2342}$  is equivalent to the third fundamental representation of  $Sp_6$ , we denote this homomorphism by  $\varpi_3$ . See [S-K], pp. 107-108 [I], §5 for some discussion of this representation. Note that on the level of matrices,  $\varpi_3$  depends on  $l$ , which will vary from case to case.

The image of  $Sp_6(\mathbb{A})$  in  $Sp_{14}(\mathbb{A})$  does not split under the covering map  $\widetilde{Sp}_{14}(\mathbb{A}) \rightarrow Sp_{14}(\mathbb{A})$ . Thus an element of  $\Theta_7^\psi$  is a genuine function defined on the double cover of  $C(\mathbb{A}) \cong Sp_6(\mathbb{A})$ , which we denote  $\widetilde{C}(\mathbb{A})$ . In addition let  $\widetilde{F}_4(\mathbb{A})$  denote the metaplectic double cover of  $F_4(\mathbb{A})$  defined using the Matsumoto cocycle. Then  $U(\mathbb{A}) \rtimes \widetilde{C}(\mathbb{A}) \hookrightarrow \widetilde{F}_4(\mathbb{A})$ .

The options in this case were discussed in detail in section 4. In brief, since each vector  $\tilde{\theta}_\phi^\psi$  in the space of  $\Theta_7^\psi$  restricts to a genuine function on  $\widetilde{Sp}_6(\mathbb{A})$ , our global integral should include

either an Eisenstein series which is genuine and a cusp form which is not, or an Eisenstein series which is not genuine and a cusp form which is. If  $P = P_2$  or  $P_3$  then  $\dim \tau = 0$ , which means that  $\tau$  is a character, and this is incompatible with  $E_\tau$  being genuine. Therefore, in these cases it must be the cusp form which is genuine. If  $P = P_1$  or  $P_4$ , then as far as we know both options are available. However, our arguments will be the same regardless of which function is genuine. For concreteness, we assume that the cusp form is genuine in these cases as well. Thus, our integral is

$$\int_{C(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U(F) \backslash U(\mathbb{A})} \tilde{\theta}_\phi^\psi(l(u)\varpi_3(g)) E_\tau(ug, s) du dg,$$

where  $\tilde{\varphi}_\pi$  is a cusp form in the space of some genuine irreducible cuspidal automorphic representation of  $\widetilde{Sp}_6(\mathbb{A})$ , and  $E_\tau$  is an Eisenstein series defined on the group  $F_4(\mathbb{A})$ , and  $\tilde{\theta}_\phi^\psi$  is a theta series from the representation  $\Theta_7^\psi$  of  $\mathcal{H}_{15}(\mathbb{A}) \rtimes \widetilde{Sp}_{14}(\mathbb{A})$ .

Referring to the unfolding process sketched above, notice that  $C$  contains every unipotent element of  $M_\mathcal{O}$ . It follows easily that for all standard parabolic subgroups  $P$  we have

$$P \backslash H / CU_\mathcal{O} = P \backslash G / P_\mathcal{O} = W(M, T) \backslash W / W(M_\mathcal{O}, T).$$

In particular, the integral is of open orbit type for all  $P$ , and one can take  $\nu_0$  to be the identity in all cases. Since  $\nu_0$  is the identity, it follows that  $L_{\nu_0} = Q_{w_0} \cap C$ . This is a parabolic subgroup of  $C$  and its Levi part is equal to the intersection of  $C$  with the Levi of  $Q_{w_0}$ .

**9.1.  $\mathbf{P} = \mathbf{P}_1$ .** This case is an example of the phenomenon discussed in subsection 5.2. In this case,  $w_0 = w[1, 2, 3, 2, 1, 4, 3, 2, 1, 3, 2, 4, 3, 2, 1]$ , while  $Q_{w_0} = M_\mathcal{O} = M_1$ . As already mentioned, the integral (14) is of open orbit type for this Fourier coefficient, regardless of the parabolic subgroup used to form the Eisenstein series. Further,  $\nu_0 = \text{identity}$ ,  $L_{\nu_0} = M_\mathcal{O} = M_1$ , and  $V$  is trivial. Hence,  $I_{w_0, \nu_0}$  is given by

$$\int_{C(F) \backslash C(\mathbb{A})} \int_{U_\mathcal{O}(\mathbb{A})} f_\tau(w_0 gu, s) \tilde{\varphi}_\pi(g) \tilde{\theta}_\phi^\psi(\varpi_3(g)l(u)) du dg,$$

while the inner period (52) is given by

$$\int_{C(F) \backslash C(\mathbb{A})} \varphi_\tau(g) \tilde{\varphi}_\pi(g) \tilde{\theta}_\phi^\psi(g) dg.$$

**Lemma 53.** Conjecture 31 is satisfied in this case.

*Proof.* The space  $P \backslash H / CU_\mathcal{O} = P \backslash H / P_\mathcal{O}$  is represented by

$$e, w[1], w[1, 2, 3, 2, 1], w[1, 2, 3, 2, 4, 3, 2, 1], \text{ and } w_0.$$

Let  $w$  denote one of these representatives, and let

$$I_w = \int_{Q_w^0(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U_{\mathcal{O}}^w(F) \backslash U_{\mathcal{O}}(\mathbb{A})} f_\tau(wug, s) \tilde{\theta}_\phi^\psi(l(u)\varpi_3(g)) du dg.$$

Here  $Q_w^0 = Sp_6 \cap w^{-1}Pw$ . We must show that  $I_w = 0$  for all  $w \neq w_0$ .

If  $w = e$  or  $w[1]$ , then  $w \cdot 2342 \in \Phi(U, T)$  which means that  $f_\tau(wug, s)$  is invariant by  $U_{2342}(\mathbb{A})$  on the left. This clearly forces the integral  $I_w$  to be zero.

If  $w = w[1, 2, 3, 2, 1]$ , then  $Q_w^0$  is the maximal parabolic subgroup of  $Sp_6$  with Levi isomorphic to  $Sp_4 \times GL_1$ . Fix an isomorphism of  $U_{\mathcal{O}}$  with  $\mathcal{H}_{15}$  by ordering the roots as

$$1000, 1100, 1110, 1120, 1220; \quad 1111, 1121 | 1221, 1231; \quad 1122, 1222, 1232, 1242, 1342.$$

Here, extra spacing has been used to reflect the way that  $L_{\mathcal{O}}^{(1)}$  decomposes as a direct sum of three invariant subspaces under the action of the standard Levi subgroup of  $Q_w$ . Also, the middle of the list is marked with a vertical bar. This will be standard practice.

Note that the three  $M_{\mathcal{O}}$ -invariant subspaces actually form a flag which is preserved by the action of the unipotent radical of  $Q_w$ . Thus, if  $R$  is the maximal standard parabolic of  $Sp_{14}$  which has Levi isomorphic to  $GL_5 \times Sp_4$ , and if  $\varpi_3$  is defined by ordering the roots as above, then  $\varpi_3$  will map the Levi of  $Q_w$  into the Levi of  $R$  and the unipotent radical of  $Q_w$  into the unipotent radical of  $R$ .

Then  $w\alpha \in \Phi(U, T)$  for all of the last five roots. Therefore, by lemma 48,

$$I_w = \int_{Q_w^0(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U_{\mathcal{O}}^w(F) V_1(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} f_\tau(wug, s) \tilde{\theta}_{\phi|_2}^\psi(l(u)\varpi_3(g)) du dg,$$

where  $V_1$  is the five dimensional abelian unipotent subgroup of  $U$  corresponding to the last five roots listed above.

Since both  $h \mapsto f_\tau(wh)$  and  $h \mapsto \tilde{\theta}_{\phi|_2}^\psi(\varpi_3(h))$  are invariant by the unipotent radical of  $Q_w^0$ , the integral  $I_w$  vanishes by the cuspidality of  $\tilde{\varphi}_\pi$ .

Finally, suppose  $w = [1, 2, 3, 2, 4, 3, 2, 1]$ . Then  $Q_w^0$  is the maximal standard parabolic subgroup of  $Sp_6$  whose Levi subgroup is isomorphic to  $GL_3$ . In order to analyze  $I_w$  in this case, we shall use a different isomorphism  $U \rightarrow \mathcal{H}_{15}$ , obtained by ordering the roots as follows:

$$(54) \quad 1000; \quad 1100, 1110, 1111, 1120, 1121, 1122 | \quad 1220, 1221, 1222, 1231, 1232, 1242; \quad 1342.$$

Thus

$$\begin{aligned} l(x_{1000}(r_1)x_{1100}(r_2), \dots, x_{1122}(r_7)) &= (r_1, r_2, r_3, r_4, r_5, r_6, r_7 | 0|0) & l(x_{2342}(z)) &= (0|0|z), \\ l(x_{1220}(y_1)x_{1221}(y_2)x_{1222}(y_3)x_{1231}(y_4)x_{1232}(y_5)x_{1242}(y_6)x_{1342}(y_7)) \\ &= (0|y_1, -2y_2, y_3, 2y_4, -2y_5, y_6, -y_7 | 0). \end{aligned}$$

Since  $w \cdot 1342 = 1342 \in \Phi(U, T)$ , lemma 48 implies that

$$I_w = \int_{Q_w^0(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U_{\mathcal{O}}^w(F) U_{1342}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} f_\tau(wug, s) \tilde{\theta}_{\phi|_6}^\psi(l(u)\varpi_3(g)) du dg.$$

Next we unfold the partial theta function

$$\tilde{\theta}_{\phi|_6}^\psi(h) = \sum_{\xi \in F^6} [\omega_\psi(h)\phi](0, \xi).$$

The standard Levi subgroup of  $Q_w^0$ , isomorphic to  $GL_3$ , acts on  $F^6$  (which may be identified with the abelian subgroup of  $U_{\mathcal{O}}$  corresponding to the six roots 1100, ..., 1122) by a representation which is equivalent to the symmetric square representation of  $GL_3$ . Choose a collection  $S$  of orbit representatives, and for each representative  $\xi$  let  $O_\xi$  denote the stabilizer in  $GL_3$ . Let  $N$  denote the unipotent radical of  $Q_w^0$ . Then we have

$$I_w = \sum_{\xi \in S} \int_{O_\xi(F)N(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U_{\mathcal{O}}^w(F) U_{1342}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} [\omega_\psi(l(u)\varpi_3(g))\phi](0, \xi) f_\tau(wug, s) du dg.$$

Let  $N'(F)$  denote the six-dimensional abelian subgroup of  $U_{\mathcal{O}}(\mathbb{A})$  corresponding to the roots 1220, 1221, 1222, 1231, 1232, 1242. Then  $wN'w^{-1} = N$ . On the other hand,  $wNw^{-1}$  lies in  $U$ . Factoring the integration on  $N$  and  $N'$ , we find that

$$I_w = \sum_{\xi \in S} \int_{O_\xi(F)N(\mathbb{A}) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi^{(N, \psi_\xi)}(g) \int_{U_{\mathcal{O}}^w(F)N'(\mathbb{A})U_{1342}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} [\omega_\psi(l(u)\varpi_3(g))\phi](0, \xi) f_\tau^{(N, \psi'_\xi)}(wug, s) du dg,$$

where

$$\tilde{\varphi}_\pi^{(N, \psi_\xi)}(g) = \int_{N(F) \backslash N(\mathbb{A})} \varphi_\pi(n) \psi_\xi(n) dn, \quad f_\tau^{(N, \psi'_\xi)}(wug, s) = \int_{N(F) \backslash N(\mathbb{A})} f_\tau(nwug, s) \psi'_\xi(n) dn,$$

and  $\psi_\xi$  and  $\psi'_\xi$  are two characters of  $N$  which depend on  $\xi \in F^6$ . We need to calculate this dependence precisely.

First, take

$$n'(y) = x_{1220}(y_1)x_{1221}(y_2)x_{1222}(y_3)x_{1231}(y_4)x_{1232}(y_5)x_{1242}(y_6)$$

Then

$$l(n'(y)) = (0|y_1, -2y_2, y_3, 2y_4, -2y_5, y_6, -y_7|0),$$

so

$$[\omega_\psi(n'(y)h)\phi_1](0, \xi) = \psi(y_t\xi) = \psi(y_1\xi_7 - 2y_2\xi_6 + y_3\xi_5 + 2y_4\xi_4 - 2y_5\xi_3 + y_6\xi_2),$$

for  $\xi = (\xi_2, \dots, \xi_7) \in F^6$ . On the other hand

$$wn'(y)w^{-1} = x_{0100}(y_1)x_{0110}(y_2)x_{0120}(y_3)x_{0111}(y_4)x_{0121}(y_5)x_{0122}(y_6),$$



which is identified with

$$\begin{pmatrix} I_3 & Y \\ & I_3 \end{pmatrix} \in Sp_6, \quad Y = \begin{pmatrix} y_4 & -y_5 & y_6 \\ -y_2 & y_3 & -y_5 \\ y_1 & -y_2 & y_4 \end{pmatrix}.$$

Thus

$$\psi'_\xi \begin{pmatrix} I_3 & Y \\ & I_3 \end{pmatrix} = \psi(\text{Tr}(\Xi \cdot Y)), \quad \text{where } \Xi = \begin{pmatrix} \xi_4 & \xi_6 & \xi_7 \\ \xi_3 & \xi_5 & \xi_6 \\ \xi_2 & \xi_3 & \xi_4 \end{pmatrix}.$$

In order to describe  $\psi_\xi$ , one needs to compute the restriction of  $\varpi_3$  to  $N$ . Let

$$e'_{ij} = e_{ij} - e_{15-j, 15-i}, \quad e''_{ij} = e_{ij} + e_{15-j, 15-i}.$$

Here  $e_{ij}$  denotes the  $14 \times 14$  matrix with a 1 at  $i, j$  and 0's everywhere else. Thus  $Sp_{14}$  contains  $I_{14} + re'_{ij}$  if  $1 \leq i, j \leq 7$ ,  $I_{14} + re''_{ij}$  if  $1 \leq i \leq 7 < j < 15 - i$ , and  $I_{14} + re_{ij}$  if  $i + j = 15$ . (Here,  $I_{14}$  is the  $14 \times 14$  identity matrix.) We have

$$(55) \quad \begin{aligned} \varpi_3(x_{0110}(r)) &= I_{14} - re'_{13} - r^2 e''_{18} - 2re''_{38} + 2re''_{49}, & \varpi_3(x_{0100}(r)) &= I_{14} + re'_{12} + re''_{58} - 2e_{69}, \\ \varpi_3(x_{0111}(r)) &= I_{14} - re'_{14} - r^2 e''_{1,10} + 2re''_{39} - 2re''_{4,10}, & \varpi_3(x_{0120}(r)) &= I_{14} + re'_{15} + re''_{28} - 2re_{4,11}, \\ \varpi_3(x_{0121}(r)) &= I_{14} + re'_{16} - r^2 e''_{1,13} - 2re''_{29} + 2re''_{3,11}, & \varpi_3(x_{0122}(r)) &= I_{14} + re'_{17} + re''_{2,10} - 2re_{3,12} \end{aligned}$$

It follows from (47), (55), that

$$\begin{aligned} & \omega_\psi \left( \varpi_3 \left[ \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} g \right] \right) \phi(0, \xi_2, \dots, \xi_7) \\ &= \psi(-2(\xi_4 \xi_5 - \xi_3 \xi_6)x_{1,1} + 2(\xi_2 \xi_6 - \xi_3 \xi_4)x_{1,2} - (\xi_3^2 - \xi_2 \xi_5)x_{1,3} \\ & \quad + 2(\xi_3 \xi_7 - \xi_4 \xi_6)x_{2,1} - (\xi_4^2 - \xi_2 \xi_7)x_{2,2} - (\xi_6^2 - \xi_5 \xi_7)x_{3,1}) \\ & \quad \times \omega_\psi(\varpi_3(g)) \phi(0, \xi_2, \dots, \xi_7). \end{aligned}$$

Thus  $\psi_\xi \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} = \psi(-\text{Tr}(\Xi^{\text{ad}} X))$  where  $\Xi$  is as above, and  $\Xi^{\text{ad}}$  is the matrix whose  $i, j$  entry is the determinant of the  $i, j$  minor of  $\Xi$ .

Now, if  $\Xi$  is trivial or of rank one, then  $\Xi^{\text{ad}}$  is trivial, and hence  $\tilde{\varphi}_\pi^{(N, \psi_\xi)}$  vanishes identically. On the other hand, if  $\Xi$  is of rank three, then  $f_\tau^{(N, \psi'_\xi)}$  is a Fourier coefficient of  $f_\tau$ , which is attached to the orbit  $(2^3)$ . According to the table,  $\tau$  is attached to  $(2^2 1^2)$ , so  $f_\tau^{(N, \psi'_\xi)}$  vanishes in this case.

This leaves the set of symmetric matrices of rank two. This set is a single orbit under the action of  $GL_3$ , and we may choose the matrix  $\Xi$  such that  $\xi_6 = 1$  and the rest are zero as a representative.

We claim that  $f_\tau^{(N, \psi_\xi)}$  is invariant by the group  $U_{0010}U_{0011}$  for this choice of  $\xi$ . To see this, consider the Fourier expansion

$$f_\tau^{(N, \psi_\xi)}(g, s) = \sum_{a, b \in F} f_\tau^{(N'', \psi_{\xi, a, b})}(g, s),$$

$$f_\tau^{(N'', \psi_{\xi, 0, b})}(g, s) := \int_{(F \setminus \mathbb{A})^2} f_\tau^{(N, \psi_\xi)}(x_{0010}(r_1)x_{0011}(r_2)g, s) \psi(ar_1 + br_2) dr.$$

If  $a$  and  $b$  are both nonzero, then  $f_\tau^{(N'', \psi_{\xi, a, b})}$  is a Fourier coefficient attached to the orbit (42) of  $Sp_6$ . Such a coefficient vanishes identically on the space of  $\tau$ . If one of  $a, b$  is zero and the other is nonzero, then after conjugating by  $w[3]$  if necessary, we may assume that  $a = 0$  and  $b$  is nonzero. Then one may rewrite  $f_\tau^{(N'', \psi_{\xi, a, b})}$  as an iterated integral with the inner integral being

$$f_\tau^{(V, \psi_V)}(g, s) := \int_{V(F) \setminus V(\mathbb{A})} f_\tau(vg, s) \psi_V(v) dv, \quad V = \left\{ \begin{pmatrix} I_2 & X & * \\ & I_2 & X' \\ & & I_2 \end{pmatrix} : X = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \right\} \subset Sp_6,$$

$$\psi_V \begin{pmatrix} I_2 & X & * \\ & I_2 & X' \\ & & I_2 \end{pmatrix} = \psi(bx_{1,1} + x_{2,2}).$$

Now, let  $V'$  be the group defined in the same manner as  $V$  but with  $X$  being arbitrary. Then every extension of  $\psi_V$  to a character of  $V'$  is in general position. Integration over  $V'$  against a character in general position is a Fourier coefficient attached to the orbit (3<sup>2</sup>). Such a Fourier coefficient vanishes identically on the space of  $\tau$ . From this we deduce that  $f^{(V, \psi_V)}$  vanishes identically, and thence so does  $f_\tau^{(N'', \psi_{\xi, 0, b})}$ .

This leaves only the constant term in the expansion of  $f_\tau^{(N, \psi'_\xi)}$  along  $U_{0010}U_{0011}$ , which proves that  $f_\tau^{(N, \psi'_\xi)}$  is invariant by this group. This permits us to express  $I_w$  as an iterated integral with the inner integral being the constant term of  $\tilde{\varphi}_\pi$  along the maximal parabolic subgroup of  $Sp_6$  with Levi isomorphic to  $GL_2 \times SL_2$ . Thus  $I_w$  vanishes.  $\square$

9.2.  $\mathbf{P} = \mathbf{P}_2$ . The basic data in this case is as follows:

$$\begin{aligned}
w_0 &= w[2, 3, 2, 1, 4, 3, 2, 1, 3, 2, 4, 3, 2, 1] \\
M_{w_0} &\cong GL_3 \times GL_1 \quad (\Delta \cap \Phi(M_{w_0}, T) = \{\alpha_3, \alpha_4\}), \\
\nu_0 &= \text{identity}, \\
L_{\nu_0} &= C \cap Q_{w_0} \\
&= \text{the maximal parabolic subgroup of } Sp_6 \text{ with Levi } \cong GL_3 \\
m(w_0 L_{\nu_0} w_0^{-1}) &\mapsto \left\{ \left( \begin{pmatrix} \det g^{-1} & \\ & 1 \end{pmatrix}, g \right) : g \in GL_3 \right\} \quad (M \hookrightarrow GL_2 \times GL_3) \\
U^{w_0} &= U_{1342} \\
V = U_{1000} &\mapsto \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, I_3 \right\} \quad (M \hookrightarrow GL_2 \times GL_3) \\
\dim \tau &= 0.
\end{aligned}$$

Here,  $M_{w_0}$  is the Levi factor of  $Q_{w_0}$ .

Fix an isomorphism  $l : U \rightarrow \mathcal{H}_{15}$  as described in section 8, based on ordering the roots as in (54). We consider the restriction of the corresponding embedding  $\varpi_3 : Sp_6 \rightarrow Sp_{14}$  to the parabolic subgroup  $L_{\nu_0}$ . First the Levi factor of  $L_{\nu_0}$  is isomorphic to  $GL_3$ , and we may write

$$\varpi_3(g) = \text{diag}(\det g, \text{sym}^2(g), \text{sym}^2(g)^*, \det g^{-1}),$$

where the  $6 \times 6$  matrix  $\text{sym}^2(g)$  denotes the matrix of  $g$  acting on the symmetric square representation for a suitable choice of basis.

Since  $\tau$  is a character in this case,  $f_\tau$  will be left-invariant by  $V$ . Hence, the integral  $I_{w_0, \nu_0}$  for this case is

$$\int_{U_{1342}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} \int_{(P_2 \cap C)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) f_\tau(w_0 u g, s) \int_{F \backslash \mathbb{A}} \tilde{\theta}_\phi^\psi(l(x_{1342}(r)u) \varpi_3(g)) dr dg du.$$

By lemma 48, part (i)

$$I_{w_0, \nu_0} = \int_{(P_2 \cap C)(F) \backslash C(\mathbb{A})} \int_{U_{1342}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} \tilde{\varphi}_\pi(g) f_\tau(w_0 u g, s) \tilde{\theta}_{\phi|_6}^\psi(l(u) \varpi_3(g)) du dg,$$

where  $\tilde{\theta}_{\phi|_6}^\psi$  is defined as in Lemma 48. The corresponding inner period (52) is given by

$$\begin{aligned}
&\int_{GL_3(F) \backslash GL_3(\mathbb{A})} \int_{\text{Mat}_{3 \times 3}^0(F) \backslash \text{Mat}_{3 \times 3}^0(\mathbb{A})} \tilde{\varphi}_\pi \left( \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} \begin{pmatrix} g & \\ & {}_t g^{-1} \end{pmatrix} \right) \tau(w_0 g w_0^{-1}) \\
&\quad \tilde{\theta}_{\phi|_6}^\psi \left( \varpi_3 \left[ \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} \begin{pmatrix} g & \\ & {}_t g^{-1} \end{pmatrix} \right] \right) dX dg,
\end{aligned}$$

(Keep in mind that  $\tau$  is a character.) Identify  $\text{Mat}_{3 \times 3}^0$  with the unipotent radical of  $C \cap P_2$  via the mapping  $X \mapsto \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix}$  and the isomorphism  $Sp_6 \rightarrow C$ . Also, identify  $GL_3$  with the Levi of  $C \cap P_2$  via the mapping  $g \rightarrow \begin{pmatrix} g & \\ & {}_t g^{-1} \end{pmatrix}$  and the isomorphism  $Sp_6 \rightarrow C$ .

Observe that for  $g_1 \in GL_3(F)$ ,  $\tilde{g} \in \widetilde{Sp}_{14}(\mathbb{A})$ , if  $\xi' := (\xi_2, \dots, \xi_7) \cdot \text{sym}^2(g_1)$ , then

$$\begin{aligned} \omega_\psi \left( \varpi_3 \left[ \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} \tilde{g} \right] \right) \phi(0, \xi') &= \omega_\psi \left( \varpi_3 \left[ \begin{pmatrix} g_1 & \\ & {}_t g_1^{-1} \end{pmatrix} \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} \tilde{g} \right] \right) \phi(0, \xi_2, \dots, \xi_7) \\ &= \omega_\psi \left( \varpi_3 \left[ \begin{pmatrix} I_3 & g_1 X {}_t g_1 \\ & I_3 \end{pmatrix} \begin{pmatrix} g_1 & \\ & {}_t g_1^{-1} \end{pmatrix} \tilde{g} \right] \right) \phi(0, \xi_2, \dots, \xi_7). \end{aligned}$$

Hence, if we write  $\Xi * g_1$  for the matrix analogous to  $\Xi$  corresponding to the vector  $(\xi_2, \dots, \xi_7) \cdot \text{sym}^2(g_1)$ , then we have

$$\psi(\text{Tr}((\Xi * g_1)^{\text{ad}} X)) = \psi(\text{Tr}(\Xi^{\text{ad}}(g_1 X {}_t g_1))) = \psi(\text{Tr}({}_t g_1 \Xi^{\text{ad}} g_1 X)) = \psi(\text{Tr}((g_1 \Xi {}_t g_1)^{\text{ad}} X)).$$

It follows that the polynomial identity  $\Xi * g_1 = g_1 \Xi {}_t g_1$  holds for all  $g_1 \in GL_3(F)$ ,  $\Xi \in \text{Mat}_{3 \times 3}^0(F)$ .

Now unfold the theta function, identify  $(\xi_2, \dots, \xi_7) \in F^6$  with  $\Xi \in \text{Mat}_{3 \times 3}^0(F)$ , and split the sum over  $\Xi$  up into orbits for the action of  $GL_3(F)$ . It is clear that  $\text{rank}(\Xi)$  is an invariant for this action. Define

$$\begin{aligned} I_{w_0, \nu_0, \Xi} &= \int_{\text{Stab}_\Xi(F) \text{Mat}_{3 \times 3}^0(\mathbb{A}) \backslash Sp_6(\mathbb{A})} \int_{U_{1342}(\mathbb{A}) \backslash U_1(\mathbb{A})} \int_{\text{Mat}_{3 \times 3}^0(F) \backslash \text{Mat}_{3 \times 3}^0(\mathbb{A})} \tilde{\varphi}_\pi \left( \begin{pmatrix} I_3 & X \\ & I_3 \end{pmatrix} g \right) \psi(\Xi^{\text{ad}} X) dX \\ &\quad f_\tau(w_0 u g, s) \omega_\psi(l(u) \varpi_3(g)) \phi(0, \xi_2, \dots, \xi_7) du dg, \end{aligned}$$

where  $\text{Stab}_\Xi$  denotes the stabilizer of  $\Xi$  in  $GL_3$ .

**Proposition 56.** The integral  $I_{w_0, \nu_0, \Xi}$  vanishes unless  $\Xi$  is of rank three.

*Proof.* If  $\Xi$  is trivial or of rank one, then  $\Xi^{\text{ad}}$  is trivial and the assertion is obvious. Suppose then that  $\Xi$  is of rank two. We may assume that  $\xi_2 = \xi_3 = \xi_4 = \xi_6 = 0$ , since each  $GL_3(F)$ -orbit contains elements with this property. The stabilizer  $\text{Stab}_\Xi$  then contains the two-dimensional unipotent subgroup  $U_{0010}U_{0011}$  which corresponds to

$$\left\{ \begin{pmatrix} 1 & 0 & r_2 \\ 0 & 1 & r_1 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3.$$

Since

$$\varpi_3(x_{0010}(r_1)x_{0011}(r_2)) = I + r_1 e'_{2,3} + r_2 e'_{24} + r_1^2 e'_{25} + r_1 r_2 e'_{26} + r_2^2 e'_{27} + 2r_1 e'_{35} + r_2 e'_{36} + r_1 e'_{46} + 2r_2 e'_{47},$$

it follows that  $\omega_\psi(l(u) \varpi_3(g)) \phi(0, 0, 0, 0, \xi_5, 0, \xi_7)$  is invariant by  $U_{0010}U_{0011}$ . Since  $\tau$  is a character, it follows that  $f_\tau$  is also invariant by  $U_{0010}U_{0011}$ . Further, the conditions we have placed on  $\xi$  imply that  $\text{Tr}(\Xi^{\text{ad}} X) = -\xi_5 \xi_7 x_{3,1}$ . Factoring the integration over  $U_{0010}U_{0011}(F) \backslash U_{0010}U_{0011}(\mathbb{A})$ ,

we obtain the constant term of  $\tilde{\varphi}_\pi$  along the parabolic subgroup of  $Sp_6$  with Levi  $GL_2 \times SL_2$  as in inner integral. Hence,  $I_{w_0, \nu_0, \Xi}$  vanishes in the rank two case as well.  $\square$

In view of proposition 56, we may write

$$I_{w_0, \nu_0} = \sum_{\Xi} I_{w_0, \nu_0, \Xi}$$

with the sum being over representatives for the distinct orbits of nondegenerate symmetric matrices. This is similar to an integral where  $P \times U_{\mathcal{O}}C$  has an open orbit  $\mathcal{X}$  in  $H$  such that  $\mathcal{X}(F)$  is a union of infinitely many  $P(F) \times U_{\mathcal{O}}(F)C(F)$ -orbits. Further, there is no apparent reason why any of the terms in the above sum should vanish.

For the sake of completeness, we verify conjecture 31 in this case as well.

**Lemma 57.** Conjecture 31 holds in this case.

*Proof.* There are seven elements of  $P \backslash H / CU_{\mathcal{O}}$ , represented by the Weyl elements

$$\begin{aligned} e, & w[2, 1], w[2, 3, 2, 1], w[2, 3, 2, 4, 3, 2, 1], w[2, 1, 3, 2, 4, 3, 2, 1], \\ & w[2, 1, 3, 2, 1, 3, 2, 4, 3, 2, 1], w[2, 3, 2, 1, 4, 3, 2, 1, 3, 2, 4, 3, 2, 1]. \end{aligned}$$

Define  $I_w$  as in lemma 53. The first three Weyl elements listed above map 2342 into the unipotent radical of  $U$ . It easily follows that  $I_w$  is zero for all choices of data in these cases.

To study the other cases, let  $Q_w^0 = C \cap Q_w$ , (a parabolic subgroup of  $Sp_6$ ). For  $w = [2, 3, 2, 4, 3, 2, 1]$ , the group  $Q_w^0$  has Levi isomorphic to  $GL_3$ . Identify  $U_{\mathcal{O}}$  with  $\mathcal{H}_{15}$  using (54). Then the last seven roots are all mapped into  $\Phi(U, T)$ . As in lemma 48, this kills every nonzero term in the sum over  $\xi \in F^7$  which defines  $\theta_\phi^\psi$ , leaving a function which is invariant by the full maximal unipotent subgroup of  $Sp_6$ . Factoring the integration over the unipotent radical of  $Q_w$ , we obtain the constant term of  $\tilde{\varphi}_\pi$  along this unipotent radical. This proves that  $I_w = 0$  in this case.

If  $w = w[2, 1, 3, 2, 4, 3, 2, 1]$ , then the situation is similar. Indeed,  $w \cdot \alpha$  is positive for  $\alpha = 1342, 1242, 1232, 1222, 1122, 1221, 1222$ . From the table, we know that  $\tau$  is a character in this case, and hence,  $f_\tau$  is again invariant by the seven dimensional unipotent group corresponding to these seven roots.

Finally, we consider  $w = w[2, 1, 3, 2, 1, 3, 2, 4, 3, 2, 1]$ . For this case, the standard Levi subgroup  $Q_w^0$  of  $L_w^0$  is isomorphic to  $GL_2 \times SL_2$ . Further,  $w \cdot \alpha > 0$  for  $\alpha = 1231, 1232, 1242, 1342$ . Let  $V_1$  denote the product of the groups  $U_\alpha$  for these four roots. Define  $l$  and  $\varpi_3$  by ordering the roots as follows:

$$1000, 1100, 1110, 1111; \quad 1120, 1121, 1122 | 1220, 1221, 1222; \quad 1231, 1232, 1242, 1342$$

Then using lemma 48, one obtains

$$I_w = \int_{Q_w^0(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{V_1(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} \tilde{\theta}_{\phi|_3}^\psi(l(u)\varpi_3(g)) f_\tau(ug, s) du dg,$$

further,  $g \mapsto \tilde{\theta}_{\phi|_3}^\psi(\varpi_3(g))$  is invariant by the unipotent radical of  $Q_w^0$ , as is  $g \mapsto f_\tau(wg, s)$ . Factoring the integration over this group, we obtain zero in this case as well.  $\square$

9.3. **P = P<sub>3</sub>**. The basic data for this case is

$$\begin{aligned} w_0 &= w[3, 2, 1, 4, 3, 2, 1, 3, 2, 4, 3, 2, 1] \\ M_{w_0} &\cong GL_2 \times GL_2 \quad (\Delta \cap \Phi(M_{w_0}, T) = \{\alpha_2, \alpha_4\}), \\ \nu_0 &= \text{identity}, \\ L_{\nu_0} &= C \cap Q_{w_0} \\ &= \text{the maximal parabolic subgroup of } Sp_6 \text{ with Levi } \cong GL_2 \times SL_2 \\ m(w_0\nu_0 L_{\nu_0}\nu_0^{-1}w_0^{-1}) &\mapsto \left\{ \left( \begin{pmatrix} g_1 & \\ & 1 \end{pmatrix}, g_2 \right) : g_1, g_2 \in GL_2, \det g_1 = \det g_2^{-2} \right\}, \quad (M \hookrightarrow GL_3 \times GL_2) \\ U^{w_0} &= U_{1242}U_{1342} \\ V &= U_{0100}U_{1100} \\ &\rightarrow \left\{ \left( \begin{pmatrix} 1 & x_1 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}, I_2 \right) \right\} \quad (M \hookrightarrow GL_3 \times GL_2) \end{aligned}$$

**Lemma 58.** Conjecture 31 holds in this case.

*Proof.* Define  $I_w$  as in lemma 53. The set  $P \backslash H / CU_{\mathcal{O}}$  has five elements,  $w_0$  and the following four others:

$$e, \quad w[3, 2, 1], \quad w[3, 2, 4, 3, 2, 1], \quad w[3, 2, 1, 3, 2, 4, 3, 2, 1].$$

Once again, by the table, we know that  $\tau$  is a character. The first three coset representatives above map 2342 to a positive root. It follows that  $I_w$  vanishes in all these cases.

Now assume  $w = w[3, 2, 1, 3, 2, 4, 3, 2, 1]$ . Then  $w\alpha > 0$  for  $\alpha = 1122, 1222, 1231, 1232, 1242, 1342$ . Order the roots of  $T$  in  $U_{\mathcal{O}}/(U_{\mathcal{O}}, U_{\mathcal{O}})$  so that these six roots come last. Then

$$I_w = \int_{Q_w^0(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U_{\mathcal{O}}^w(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} f_\tau(wug, s) \theta_{\phi|_1}^\psi(l(u)\varpi_3(g)) du dg,$$

and both  $g \mapsto f_\tau(wg, s)$  and  $g \mapsto \theta_{\phi|_1}^\psi(\varpi_3(g))$  are invariant by the unipotent radical of  $Q_w^0$ , which proves that  $I_w$  is zero, since  $\tilde{\varphi}_\pi$  is cuspidal.  $\square$

We now proceed to analyze  $I_{w_0}$ . Using once again the fact that  $\tau$  is a character, we obtain

$$I_{w_0, \nu_0} = \int_{U_{1242}U_{1342}(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} \int_{(P_3 \cap C)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) f_\tau(w_0 u g, s) \tilde{\theta}_{\phi|_5}^\psi(u g) dg du.$$

Define  $l : U \rightarrow \mathcal{H}_{15}$  and  $\varpi_3 : Sp_6 \rightarrow Sp_{14}$  using the ordering given in equation (54). Identify  $L_{\nu_0}$  with  $Sp_6$  using the isomorphism fixed in section 6. Recall that  $L_{\nu_0}$  is a standard parabolic subgroup of  $Sp_6$  in this case. The Levi factor of  $L_{\nu_0}$  is isomorphic to  $GL_2 \times SL_2$  and consists of all matrices of the form  $\text{diag}(g_1, g_2, {}_t g_1^{-1})$ . We identify this group with  $GL_2 \times SL_2$  via the map  $(g_1, g_2) \mapsto \text{diag}(g_1, g_2, {}_t g_1^{-1})$ . For elements of this group  $\varpi_3$  is given by

$$\text{diag}(g_1, g_2, {}_t g_1^{-1}) \mapsto \text{diag}(\det g_1 \cdot g_2, g_1, \rho(g_1, g_2), {}_t g_1^{-1}, \det g_1^{-1} \cdot {}_t g_2^{-1}), \quad (g_1 \in GL_2, g_2 \in SL_2).$$

Here,  $\rho(g_1, g_2)$  is a certain  $6 \times 6$  matrix.

Plug in

$$\tilde{\theta}_{\phi|_5}^\psi(u g) = \sum_{\xi \in F^5} [\omega_\psi(u g) \phi](0, 0, \xi_3, \dots, \xi_7),$$

and split  $\xi$  up into  $\xi' = (\xi_3, \xi_4) \in F^2$  and  $\xi'' = (\xi_5, \xi_6, \xi_7) \in F^3$ . For  $g$  in the  $GL_2$  factor of the Levi of  $L_{\nu_0}$ , we have  $\varpi_3(g) = \text{diag}(\det g, \det g, g, s^2(g), {}_t s^2(g)^{-1}, {}_t g^{-1}, \det g^{-1}, \det g^{-1})$ , where

$$s^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & bc + ad & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

Thus  $GL_2(F)$  acts on  $\{\xi' = (\xi_3, \xi_4) : \xi_3, \xi_4 \in F\}$  with two orbits. The contribution from the trivial orbit is

$$(59) \quad I_{w_0, \nu_0, 0} = \int_{U_{1242}U_{1342}(\mathbb{A}) \backslash U_1(\mathbb{A})} \int_{(P_3 \cap C)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) f_\tau(w_0 u g, s) \tilde{\theta}_{\phi|_3}^\psi(u g) dg du.$$

To see that this is equal to zero, one checks that

$$\left\{ \begin{pmatrix} I_2 & * & * \\ & I_2 & * \\ & & I_2 \end{pmatrix} \right\} \subset Sp_6 \text{ maps into } \left\{ \begin{pmatrix} I_2 & * & * & * & * \\ & I_2 & * & * & * \\ & & I_6 & * & * \\ & & & I_2 & * \\ & & & & I_2 \end{pmatrix} \right\} \subset Sp_{14},$$

under the embedding  $\varpi_3 : Sp_6 \rightarrow Sp_{14}$ , defined using the ordering given in (54). It follows from lemma 48 that the function  $\tilde{\theta}_{\phi|_3}^\psi(u g)$  is invariant by this group, and, since  $\tau$  is a character,  $f_\tau(w_0 u g, s)$  is invariant as well. It follows that the mapping  $\tilde{\varphi}_\pi \mapsto I_{w_0, \nu_0, 0}$ , where  $I_{w_0, \nu_0, 0}$  is defined by equation (59), factors through the constant term of  $\tilde{\varphi}_\pi$ , and hence vanishes.

To describe the contribution from the open orbit, let  $S_{0,1}$  denote the stabilizer of  $\xi' = (0, 1)$  in  $GL_2$  (still identified with a subgroup of  $L_{\nu_0}$  as above). It consists of all matrices of the

form  $\begin{pmatrix} \alpha & * \\ & 1 \end{pmatrix}$ . Also,  $S_{0,1} \backslash GL_2$  may be identified with  $P_2^0 \backslash L_{\nu_0}$ , where

$$P_2^0 := \left\{ \begin{pmatrix} \alpha & * & * & * & * \\ & 1 & * & * & * \\ & & h & * & * \\ & & & 1 & * \\ & & & & \alpha^{-1} \end{pmatrix} \in Sp_6 : \alpha \in GL_1, h \in SL_2 \right\}.$$

It follows that

$$\sum_{\xi' \in F^2 \setminus \{(0,0)\}} \sum_{\xi'' \in F^3} \omega_\psi(\tilde{g}) \phi_1(0, 0, \xi', \xi'') = \sum_{\gamma \in P_2^0(F) \backslash L_{\nu_0}(F)} \sum_{\xi'' \in F^3} \omega_\psi(\gamma \tilde{g}) \phi_1(0, 0, 0, 1, \xi''),$$

for all  $\tilde{g} \in \widetilde{Sp}_{14}(\mathbb{A})$  and  $\phi_1 \in S(\mathbb{A}^7)$ . Plugging in,

$$I_{w_0, \nu_0} = \int_{P_2^0(F) \backslash Sp_6(\mathbb{A})} \int_{U'(\mathbb{A})} \tilde{\varphi}_\pi(g) \sum_{\xi'' \in F^3} \omega_\psi(l(u') \varpi_3(g)) \phi(0, 0, 0, 1, \xi'') f_\tau(w_0 u' g, s) du' dg.$$

Here  $U' = U_{1242} U_{1342} \backslash U_{\mathcal{O}}$ .

Next, it follows from (47) and (55) that

$$(60) \quad \omega_\psi \left( \varpi_3 \begin{pmatrix} 1 & 0 & 0 & 0 & q & r \\ 0 & 1 & 0 & 0 & p & q \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right) \phi_1(0, 0, 0, 1, \xi'') = \psi(-p) \phi_1(0, 0, 0, 1, \xi''),$$

for all  $p, q, r \in \mathbb{A}$ , all  $\xi'' \in F^3$ , and all  $\phi_1 \in S(\mathbb{A}^7)$ .

We shall write  $Z$  for the three dimensional unipotent group considered in (60), because it is the center of the unipotent radical of  $L_{\nu_0}$ . Then  $h \mapsto f_\tau(w_0 h, s)$  is left-invariant by  $Z(\mathbb{A})$ .

It follows that  $I_{w_0, \nu_0}$  is equal to

$$(61) \quad \int_{P_2^0(F) Z(\mathbb{A}) \backslash Sp_6(\mathbb{A})} \left[ \int_{Z(F) \backslash Z(\mathbb{A})} \tilde{\varphi}_\pi(zg) \psi_Z(z) dz \right] \times \\ \int_{U'(\mathbb{A})} \left[ \sum_{\xi'' \in F^3} \omega_\psi(l(u') \varpi_3(g)) \phi(0, 0, 0, 1, \xi'') \right] f_\tau(w_0 u' g, s) du' dg,$$

where  $\psi_Z(z) = \psi(-p)$  in the coordinates of (60).



Now, the function  $\int_{Z(F)\backslash Z(\mathbb{A})} \tilde{\varphi}_\pi(zg)\psi_Z(z) dz$  is invariant by  $Y(F)$ , where

$$(62) \quad Y = \left\{ y = \begin{pmatrix} 1 & 0 & a & b & q & r \\ 0 & 1 & 0 & 0 & p & q \\ 0 & 0 & 1 & 0 & 0 & b \\ 0 & 0 & 0 & 1 & 0 & -a \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_6 \right\}$$

Hence we can apply Fourier expansion on the group  $Y(F)Z(\mathbb{A})\backslash Y(\mathbb{A})$ . The subgroup of  $P_2^0$  consisting of matrices of the form  $\text{diag}(\alpha, 1, h, 1, \alpha^{-1}) : \alpha \in GL_1, h \in SL_2$  acts on the space of characters of  $Y(F)Z(\mathbb{A})\backslash Y(\mathbb{A})$  with 2 orbits. We claim that the trivial orbit contributes zero. The term corresponding to the trivial character is

$$(63) \quad \int_{P_2^0(F)Z(\mathbb{A})\backslash Sp_6(\mathbb{A})} \int_{U'(\mathbb{A})} \int_{Y(F)\backslash Y(\mathbb{A})} \tilde{\varphi}_\pi(yg)\psi_Y^0(y) dy \\ \sum_{\xi'' \in F^3} \omega_\psi(l(u')\varpi_3(g))\phi(0, 0, 0, 1, \xi'')f_\tau(w_0u'g, s) du' dg,$$

where  $\psi_Y^0$  is the trivial extension of  $\psi_Z$  to a character of  $Y$ . In the coordinates of (62), it is given by  $\psi_Y^0(y) = \psi(-p)$ .

**Lemma 64.** The integral (63) vanishes identically.

*Proof.* In the proof we shall repeatedly use the formulae for  $\omega_\psi$  given in (46) and (47). Set  $\tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(g) = \int_{Y(F)\backslash Y(\mathbb{A})} \tilde{\varphi}_\pi(yg)\psi_Y^0(y) dy$ . Factor the integration over  $Y(F)Z(\mathbb{A})\backslash Y(\mathbb{A})$ ,

which may be identified with  $U_{0011}U_{0111}(F)\backslash U_{0011}U_{0111}(\mathbb{A})$ . Note that  $f_\tau$  and  $\tilde{\varphi}_\pi^{(Y, \psi_Y^0)}$  are both invariant by this group on the left. One computes

$$\omega_\psi(\varpi_3(x_{0111}(r_3)x_{0011}(r_2)))\phi(0, 0, 0, 1, \xi_5, \xi_6, \xi_7) = \psi(-2r_3\xi_5)\phi(0, 0, 0, 1, \xi_5, \xi_6, \xi_7 + 2r_2),$$

z deducing that

$$\begin{aligned}
& \int_{(F \setminus \mathbb{A})^2} \sum_{\xi'' \in F^3} \int_{U'(\mathbb{A})} \omega_\psi(l(u') \varpi_3(x_{0111}(r_3) x_{0011}(r_2) g)) \phi(0, 0, 0, 1, \xi_5, \xi_6, \xi_7) \\
& \quad f_\tau(w_0 u' x_{0111}(r_3) x_{0011}(r_2) g, s) \tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(x_{0111}(r_3) x_{0011}(r_2) g) du' dr \\
(65) \quad & = \int_{(F \setminus \mathbb{A})^2} \sum_{\xi'' \in F^3} \int_{U'(\mathbb{A})} \omega_\psi(\varpi_3(x_{0111}(r_3) x_{0011}(r_2)) l(u') \varpi_3(g)) \phi(0, 0, 0, 1, \xi_5, \xi_6, \xi_7) \\
& \quad f_\tau(w_0 x_{0111}(r_3) x_{0011}(r_2) u' g, s) \tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(x_{0111}(r_3) x_{0011}(r_2) g) du' dr \\
& = \int_{(F \setminus \mathbb{A})^2} \sum_{\xi'' \in F^3} \int_{U'(\mathbb{A})} \psi(-2r_3 \xi_5) \omega_\psi(l(u') \varpi_3(g)) \phi(0, 0, 0, 1, \xi_5, \xi_6, \xi_7 + 2r_2) \\
& \quad f_\tau(w_0 x_{0111}(r_3) x_{0011}(r_2) u' g, s) \tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(x_{0111}(r_3) x_{0011}(r_2) g) du' dr.
\end{aligned}$$

As indicated in the table above, in this case  $\tau$  is a character. Since  $w_0 \cdot 0111$  and  $w_0 \cdot 0011$  are both positive, it follows that  $f_\tau(w_0 x_{0111}(r_3) x_{0011}(r_2) u' g, s) = f_\tau(w_0 u' g, s)$ . Also it follows from the definition of  $\tilde{\varphi}_\pi^{(Y, \psi_Y^0)}$  that  $\tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(x_{0111}(r_3) x_{0011}(r_2) g) = \tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(g)$ . Collapsing the summation on  $\xi_7$  with the integration on  $r_2$ , and applying Fourier inversion in  $r_3$  and  $\xi_5$ , we find that (65) equals

$$\int_{U'(\mathbb{A})} \int_{\mathbb{A}} \sum_{\xi_6 \in F} \omega_\psi(l(u') \varpi_3(g)) \phi(0, 0, 0, 1, 0, \xi_6, r_2) dr_2 f_\tau(w_0 u' g, s) \tilde{\varphi}_\pi^{(Y, \psi_Y^0)}(g) du'$$

Next,  $\varpi_3(x_{0001}(r)) = I + r e'_{34} + r e'_{56} + 2r e'_{67} + r^2 e'_{57}$ , hence  $\omega_\psi(\varpi_3(x_{0001}(r))) \phi(0, 0, 0, 1, 0, \xi_6, r_2) = \phi(0, 0, 0, 1, 0, \xi_6, r_2 - 2r \xi_6)$ , and so

$$\int_{U'(\mathbb{A})} \int_{\mathbb{A}} \sum_{\xi_6 \in F} \omega_\psi(l(u') \varpi_3(g)) \phi(0, 0, 0, 1, 0, \xi_6, r_2) dr_2 f_\tau(w_0 u' g, s) du'$$

is invariant by  $U_{0001}$ . Factoring the integration, one obtains an expression for (63) as an iterated integral, such that the inner integration is the constant term of  $\tilde{\varphi}_\pi$  along the unipotent radical of the parabolic subgroup of  $Sp_6$  whose Levi part is  $GL_1 \times Sp_4$ . It then follows that the integral (63) is zero.  $\square$

We continue to study the expansion of (61) along  $Y(F) \setminus Y(\mathbb{A})$ . Next we consider the contribution from the nontrivial orbit. Choose the character  $\psi_Y(y) = \psi(a - p)$  (in terms of the coordinates given in (62)) as a representative for this orbit. Then the stabilizer inside  $GL_1 \times SL_2$  as embedded in  $Sp_6$  is the semidirect product of  $T_1 = \text{diag}(\alpha, 1, \alpha, \alpha^{-1}, 1, \alpha^{-1})$ ,

and the unipotent group  $I_6 + re_{3,4}$ . Thus the integral (61) is equal to

$$(66) \quad \int_{Z(\mathbb{A})T_1(F)U_{\max}^{Sp_6}(F)\backslash Sp_6(\mathbb{A})} \int_{U'(\mathbb{A})} \int_{Y(F)\backslash Y(\mathbb{A})} \tilde{\varphi}_\pi(yg)\psi_Y(y) dy \\ \sum_{\xi'' \in F^3} \omega_\psi(l(u')\varpi_3(g))\phi(0,0,0,1,\xi'')f_\tau(w_0u'g,s) du' dg.$$

Here,  $U_{\max}^{Sp_6}$  is the standard maximal unipotent subgroup of  $Sp_6$ . Next we factor the integration over  $U_{0111}(F)\backslash U_{0111}(\mathbb{A})$ . As a subgroup of  $Sp_6$  this group is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & m & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & m \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Note that

$$\tilde{\varphi}_\pi^{(Y,\psi_Y)}(g) := \int_{Y(F)\backslash Y(\mathbb{A})} \tilde{\varphi}_\pi(yg)\psi_Y(y) dy$$

is invariant by this group, as is  $f_\tau$ . Using (55), one can check that

$$\omega_\psi(\varpi_3(x_{0111}(m)))\phi(0,0,0,1,\xi_5,\xi_6,\xi_7) = \psi(-2m\xi_5)\phi(0,0,0,1,\xi_5,\xi_6,\xi_7).$$

Integration over  $m$  then picks of the term  $\xi_5 = 0$ . Hence (66) equals

$$\int_{Z_1(\mathbb{A})T_1(F)U_{\max}^{Sp_6}(F)\backslash Sp_6(\mathbb{A})} \int_{U'(\mathbb{A})} \tilde{\varphi}_\pi^{(Y,\psi_Y)}(g) \sum_{\xi'' \in F^2} \omega_\psi(l(u')\varpi_3(g))\phi(0,0,0,1,0,\xi'')f_\tau(w_0u'g,s) du' dg,$$

where  $Z_1 = U_{0111}Z \subset Y$ . Next, we consider the group  $L := U_{0010}U_{0011}$ . We compute

$$\varpi_3(x_{0010}(\xi_6)x_{0011}(\xi_7/2)) = \begin{pmatrix} h & \\ & {}_t h^{-1} \end{pmatrix}, \quad \text{where } h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & \xi_6 & \xi_7/2 & \xi_6^2 & \xi_6\xi_7/2 & \xi_7^2/4 \\ & & 1 & 0 & 2\xi_6 & \xi_7/2 & 0 \\ & & & 1 & 0 & \xi_6 & \xi_7 \\ & & & & 1 & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix},$$

Hence, by (47),

$$\sum_{\xi_6, \xi_7 \in F} \omega_\psi(\varpi_3(g))\phi(0,0,0,1,0,\xi_6,\xi_7) = \sum_{\gamma \in U_{0010}U_{0011}(F)} \omega_\psi(\varpi_3(\gamma g))\phi(\xi_0),$$

where  $\xi_0 = (0,0,0,1,0,0,0)$ . Collapsing summation with integration, the above integral is equal to

$$(67) \quad \int_{Z_1(\mathbb{A})T_1(F)V_1(F)\backslash Sp_6(\mathbb{A})} \int_{U'(\mathbb{A})} \tilde{\varphi}_\pi^{(Y,\psi_Y)}(g)\omega_\psi(l(u')\varpi_3(g))\phi(\xi_0)f_\tau(w_0u'g,s) du' dg,$$

where

$$V_1 = \left\{ \begin{pmatrix} 1 & a & 0 & b & c & d \\ & 1 & 0 & e & f & * \\ & & 1 & g & * & * \\ & & & \ddots & & \end{pmatrix} \in Sp_6 \right\}$$

(a subgroup of  $U_{\max}^{Sp_6}$  complementary to  $U_{0010}U_{0011}$ ). Here, and in what follows, we exploit the fact that an element  $u$  of  $U_{\max}^{Sp_6}$  is determined by the entries  $u_{ij}$  with  $1 \leq i < j \leq 7 - i$ . Now, it is easily checked that the function  $h \mapsto f_\tau(w_0 h)$  ( $h \in F_4(\mathbb{A})$ ) is left  $V_1(\mathbb{A})$  invariant. (Here, we identify  $V_1$  with a subgroup of  $F_4$  using the identification of  $M_4 \subset F_4$  with  $GSp_6$  given in section 6.) Using (47) and (55) one can check that the function  $h \mapsto \omega_\psi(h)\phi(\xi_0)$  ( $h \in \widetilde{Sp}_6(\mathbb{A})$ ) is invariant by the groups  $U_{0001}(\mathbb{A})$ ,  $U_{0100}(\mathbb{A})$ , and  $U_{0110}(\mathbb{A})$ . It then follows that

$$g \mapsto \int_{U'(\mathbb{A})} f_\tau(w_0 u' g, s) \omega_\psi(l(u') \varpi_3(g)) \phi(\xi_0) du'$$

is also invariant by  $U_{0001}(\mathbb{A})$ ,  $U_{0100}(\mathbb{A})$ , and  $U_{0110}(\mathbb{A})$ . The product of these three groups may be identified with  $Z_1(\mathbb{A}) \backslash V_1(\mathbb{A})$ . Therefore, when we factor the integration over  $V_1(F)Z_1(\mathbb{A}) \backslash V_1(\mathbb{A})$ , the integral (67) is equal to

$$\int_{T_1(F)V_1(\mathbb{A}) \backslash Sp_6(\mathbb{A})} \int_{Y_2(F) \backslash Y_2(\mathbb{A})} \tilde{\varphi}_\pi(y_2 g) \psi_{Y_2}(y_2) dy_2 \int_{U'(\mathbb{A})} \omega_\psi(u' \varpi_3(g)) \phi(\xi_0) f_\tau(w_0 u' g) du' dg,$$

where

$$Y_2 := \left\{ \begin{pmatrix} 1 & a & h & b & c & d \\ & 1 & 0 & e & f & * \\ & & 1 & m & * & * \\ & & & \ddots & & \end{pmatrix} \in Sp_6 \right\}, \quad \psi_{Y_2} \begin{pmatrix} 1 & a & h & b & c & d \\ & 1 & 0 & e & f & * \\ & & 1 & m & * & * \\ & & & \ddots & & \end{pmatrix} = \psi(h - f).$$

Using the left invariance property of  $\tilde{\varphi}_\pi$  under elements in  $Sp_6(F)$ , we deduce that

$$\begin{aligned} \tilde{\varphi}_\pi^{(Y_2, \psi_{Y_2})}(g) &:= \int_{Y_2(F) \backslash Y_2(\mathbb{A})} \tilde{\varphi}_\pi(y_2 g) \psi_{Y_2}(y_2) dy_2 \\ &= \int_{Y_2(F) \backslash Y_2(\mathbb{A})} \tilde{\varphi}_\pi(y_2 w[3]g) \psi'_{Y_2}(y_2) dy_2, \end{aligned}$$

where  $w[3]$  is our standard representative in  $N_G(T)$  for the simple reflection  $s_{\alpha_3} \in W(G, T)$ , and  $\psi'_{Y_2}(y_2) = \psi_{Y_2}(w[3]^{-1}y_2w[3])$ . One checks that

$$\psi'_{Y_2} \begin{pmatrix} 1 & a & h & b & c & d \\ & 1 & 0 & e & f & * \\ & & 1 & g & * & * \\ & & & \ddots & & \end{pmatrix} = \psi(a - g).$$

Finally, the function  $\tilde{\varphi}_\pi^{(Y_2, \psi'_{Y_2})}$  is invariant by  $U_{0010}(F)$ . We plug in the Fourier expansion along this group. The term corresponding to the trivial character vanishes by cuspidality.

The remaining characters are permuted transitively by the action of a one dimensional torus which is conjugate under  $w[3]$  to the torus  $T_1$ . Thus, in the end we obtain

$$\int_{V_1(\mathbb{A}) \backslash Sp_6(\mathbb{A})} \int_{U'(\mathbb{A})} W_{\tilde{\varphi}_\pi}(w[3]g) \omega_\psi(u' \varpi_3(g)) \phi(\xi_0) f_\tau(w_0 u' g) du' dg.$$

We conclude that this case is preWhittaker.

9.4. **P = P<sub>4</sub>**. The basic data in this case is

$$\begin{aligned} w_0 &= w[4, 3, 2, 1, 3, 2, 4, 3, 2, 1] \\ M_{w_0} &\cong GL_1 \times GSp_4 \quad (\Delta \cap \Phi(M_{w_0}, T) = \{\alpha_2, \alpha_3\}), \\ \nu_0 &= \text{identity}, \\ L_{\nu_0} &= C \cap Q_{w_0} \\ &= \text{the maximal parabolic subgroup of } Sp_6 \text{ with Levi } \cong GL_1 \times Sp_4 \\ &= Sp_4^{\{\alpha_2, \alpha_3\}} \cdot \alpha_4^\vee(GL_1) \\ m(w_0 L_{\nu_0} w_0^{-1}) &= Sp_4^{\{\alpha_2, \alpha_3\}} \cdot 2321^\vee(GL_1) \subset M \cong GSpin_7 \\ U^{w_0} &= U_{1122} U_{1222} U_{1232} U_{1242} U_{1342} \\ V &= U_{1000} U_{1100} U_{1110} U_{1120} U_{1220} \\ &= \text{the unipotent radical of the maximal parabolic subgroup of} \\ &M \cong GSpin_7, \text{ with Levi } \cong GL_1 \times GSpin_5. \end{aligned}$$

Here  $2321^\vee$  denotes the cocharacter of  $T$  given by

$$2321^\vee(t) = \alpha_1^\vee(t^2) \alpha_2^\vee(t^3) \alpha_3^\vee(t^2) \alpha_4^\vee(t), \quad t \in \mathbb{G}_m.$$

**Lemma 68.** Conjecture 31 holds in this case.

*Proof.* The set  $P \backslash H / CU_{\mathcal{O}}$  has only three elements, represented by the identity,  $w_0$ , and  $w := [4, 3, 2, 1]$ . The identity maps 2342 into  $U$  which proves that the term corresponding to this double coset is zero. Let  $Q_w^0 = Sp_6 \cap w^{-1} P w$ . It is the standard maximal parabolic subgroup of  $Sp_6$  with Levi subgroup isomorphic to  $GL_2 \times SL_2$ .

Fix the homomorphisms  $l : U_{\mathcal{O}} \rightarrow \mathcal{H}_{15}$  and  $\varpi_3 : Sp_6 \rightarrow Sp_{14}$  by ordering the roots of  $U_{\mathcal{O}} / (U_{\mathcal{O}}, U_{\mathcal{O}})$  as follows

$$1000, 1100; \quad 1110, 1111; \quad 1120, 1220, 1121 | 1221, 1122, 1222; \quad 1231, 1231; \quad 1242, 1342.$$

The last four roots are sent to  $\Phi(U, T)$  by  $w$ . Write  $V_w$  for the corresponding unipotent subgroup. Then applying lemma 48 we have

$$I_w = \int_{Q_w^0(F) \backslash Sp_6(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{U_{\mathcal{O}}^w(F) V_w(\mathbb{A}) \backslash U_{\mathcal{O}}(\mathbb{A})} f_\tau(wug, s) \tilde{\theta}_{\phi|_3}^\psi(l(u) \varpi_3(g)) du dg.$$

Further, both  $g \mapsto f_\tau(wg, s)$  and  $g \mapsto \tilde{\theta}_{\phi|_3}^\psi(\varpi_3(g))$  are invariant by the unipotent radical of  $Q_w^0$ , so  $I_w$  vanishes by cuspidality.  $\square$

To analyze  $I_{w_0, \nu_0}$ , it is more convenient to define the isomorphism  $l : U \rightarrow \mathcal{H}_{15}$ , and the embedding  $\varpi_3 : Sp_6 \rightarrow Sp_{14}$  by ordering of the roots of  $T$  in  $U/(U, U)$ , thus:

$$1000, 1100, 1110, 1120, 1220; \quad 1111, 1121|1221, 1231; \quad 1122, 1222, 1232, 1242, 1342.$$

(Recall that changing the isomorphism  $l$  changes the bijection  $\phi \rightarrow \tilde{\theta}_\phi^\psi$  between Schwartz functions and theta functions, but not the space of functions obtained.) It follows that

(69)

$$l^{-1}(0|y|0) = x_{1221} \left( -\frac{y_1}{2} \right) x_{1231} \left( \frac{y_2}{2} \right) x_{1122}(-y_3) x_{1222}(y_4) x_{1232} \left( -\frac{y_5}{2} \right) x_{1242}(y_6) x_{1342}(-y_7),$$

for  $y = (y_1, y_2, \dots, y_7)$ . Having identified  $C$  with  $Sp_6$ , the group  $C \cap M_2$  is identified with  $\{\text{diag}(\alpha, g, \alpha^{-1}) \subset Sp_6 : \alpha \in GL_1, g \in Sp_4\}$ . If  $g \in Sp_4$ , we may write  $\varpi_3(g) = \text{diag}(\wedge_0^2(g), g', {}_t(\wedge_0^2(g)))$ , where  $\wedge_0^2(g)$  denotes matrix for  $g$  acting on the five-dimensional fundamental representation of  $Sp_4$  with respect to a suitable basis, and  $g' = tgt^{-1}$ , for  $t = \text{diag}(-2, -2, 1, 1)$ .

We may then write

$$I_{w_0, \nu_0} = \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{(C \cap P_2)(F) \backslash C(\mathbb{A})} \int_{(F \backslash \mathbb{A})^5} \tilde{\varphi}_\pi(g) \tilde{\theta}_\phi^\psi((0|0, y|0)u_0 \varpi_3(g)) f_\tau(v(y)w_0 u_0 g, s) dy dg du_0,$$

where

$$(70) \quad v(-y_1, -y_2, -y_3, -y_4, -y_5) = x_{1000}(y_1) x_{1100}(y_2) x_{1110} \left( \frac{y_3}{2} \right) x_{1120}(-y_4) x_{1220}(y_5)$$

satisfies  $l(w_0^{-1}v(y)w_0) = (0|0, 0, y|0) \in \mathcal{H}_{15}$ , for  $y = (y_1, \dots, y_5)$ . The image of  $v(y)$  under the isomorphism  $M \rightarrow \text{GSpin}_7$  fixed in section 6 and the natural homomorphism from  $\text{GSpin}_7$  onto  $SO_7$  is

$$\begin{pmatrix} 1 & y_1 & y_2 & y_3/2 & y_4 & y_5 & * \\ & 1 & & & & & -y_5 \\ & & 1 & & & & -y_4 \\ & & & 1 & & & -y_3/2 \\ & & & & 1 & & -y_2 \\ & & & & & 1 & -y_1 \\ & & & & & & 1 \end{pmatrix} \in SO_7.$$

We now expand  $f_\tau$  along the group  $V$ . The group  $Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(GL_1(F))$  acts on the characters of  $V(F) \backslash V(\mathbb{A})$  with infinitely many orbits. Indeed, the space of characters of  $V(F) \backslash V(\mathbb{A})$  may be identified with the  $F$  points of the rational representation of  $Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(GL_1(F))$  which is dual to the quotient of  $V$  by its commutator subgroup  $(V, V)$ . We denote this space by  $V/(V, V)^*$ . Then  $2321^\vee(GL_1)$  acts linearly on  $V/(V, V)^*$ , and the action of  $Sp_4^{\{\alpha_2, \alpha_3\}}$  on  $V/(V, V)^*$  can be identified with the action of  $SO_5$  on its

standard representation (regarding  $Sp_4$  as  $Spin_5$ ). Identify  $V/(V, V)^*$  with column vectors and let  $Q$  denote the nondegenerate  $Spin_5$ -invariant quadratic form on this space. Then the orbits for  $Sp_4 \cdot GL_1$  are as follows: one orbit consists of the zero vector, and a second corresponds to all vectors  $x \neq 0$  such that  $Q(x) = 0$ . The vectors  $x$  with  $Q(x) \neq 0$  split into infinitely many orbits parametrized by square classes in  $F^\times$ . That is, two vectors  $x_1, x_2$  with  $Q(x_1)Q(x_2) \neq 0$  lie in the same orbit if and only if  $Q(x_1)/Q(x_2)$  is a square. The zero vector of course corresponds to the trivial character. As representatives for the remaining orbits we take

$$\psi_V^0(v(y)) = \psi(y_1), \quad \psi_V^1(v(y)) = \psi(y_3), \quad \text{and} \quad \{\psi_V^a(v(y)) = \psi(y_2 - ay_4)\},$$

where  $a$  ranges over a set of representatives for the nonzero, nonsquare square classes in  $F$ . (If  $a$  is a square, then  $\psi_V^a$ , defined as above, is equivalent to  $\psi_V^1$ .) The stabilizer of  $\psi_V^0$  in  $Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(F^\times)$  is  $SL_2^{\alpha_3}(F) \cdot 2421^\vee(F^\times)U_{0100}U_{0110}U_{0120}(F)$ , with the cocharacter  $2421^\vee$  being defined in the same way that  $2321^\vee$  was defined above. The space  $SL_2^{\alpha_3}(F) \cdot 2421^\vee(F^\times)U_{0100}U_{0110}U_{0120}(F) \setminus Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(F^\times)$  can be identified with  $P_3^0(F) \setminus (C \cap P_2)(F)$ , where

$$P_3^0 \mapsto \left\{ \begin{pmatrix} a & * & * & * & * \\ & a & * & * & * \\ & & h & * & * \\ & & & a^{-1} & * \\ & & & & a^{-1} \end{pmatrix} a \in GL_1, h \in SL_2 \right\} \subset Sp_6.$$

The stabilizer of  $\psi_V^1$  in  $Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(F^\times)$  is isomorphic to the semidirect product of  $SL_2 \times SL_2$  and a group of order 2 which acts by reversing the factors. Specifically, the stabilizer contains  $SL_2^{\alpha_2}, SL_2^{0120}$ , and suitable (nonstandard) representatives for the simple reflection  $w[3]$  in  $W(G, T)$ .

If  $a$  is not square, then the stabilizer of  $\psi_V^a$  in  $Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(F^\times)$  also has two components, but the identity component is isomorphic not to  $SL_2^2$ , but to  $\text{Res}_{F(\sqrt{a})/F} SL_2$ . Here  $F(\sqrt{a})$  denotes the unique quadratic extension of  $F$  in which  $a$  is a square, and  $\text{Res}_{F(\sqrt{a})/F}$  denotes restriction of scalars. If we identify  $M_1$  with  $GSp_6$  as in section 6, then  $Sp_4^{\{\alpha_2, \alpha_3\}}(F) \cdot 2321^\vee(F^\times)$  corresponds to all matrices of the form

$$\alpha \cdot \begin{pmatrix} 1 & & & & & \\ & g & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & 1 \end{pmatrix}, \quad \alpha \in GL_1, g \in Sp_4,$$

while the identity component of the stabilizer of  $\psi_V^a$  consists of all matrices of the form

$$\begin{pmatrix} 1 & & & \\ & g & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ g_2 a & g_1 & g_4 a & g_3 \\ g_5 & g_6 & g_7 & g_8 \\ g_6 a & g_5 & g_8 a & g_7 \end{pmatrix}, \quad \begin{aligned} g_1 g_7 + a g_2 g_8 - g_3 g_5 - a g_4 g_6 &= 1, \\ g_1 g_8 + g_2 g_7 - g_3 g_6 - g_4 g_5 &= 0. \end{aligned}$$

A representative for the second connected component of the stabilizer in this case is

$$\text{diag}(-1, -1, 1, 1, -1, -1),$$

which acts on the identity component in a manner which corresponds to the mapping  $h \mapsto {}_t\bar{h}^{-1}$ , with  ${}^-$  being the action of the nontrivial element of  $\text{Gal}(F(\sqrt{a})/F)$ .

From this discussion, it follows that

$$I_{w_0, \nu_0} = I_{w_0, \nu_0}^{00} + I_{w_0, \nu_0}^0 + I_{w_0, \nu_0}^1 + \sum_{\substack{a \in F^\times / F^{\times, 2} \\ a \neq 1}} I_{w_0, \nu_0}^a,$$

where

$$I_{w_0, \nu_0}^{00} = \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{(C \cap P_2)(F) \backslash C(\mathbb{A})} \int_{(F \backslash \mathbb{A})^5} \tilde{\varphi}_\pi(g) \tilde{\theta}_\phi^\psi((0|0, y|0)u_0 \varpi_3(g)) f_\tau^V(v(y)w_0 u_0 g, s) dy dg du_0,$$

$$f_\tau^V(h, s) = \int_{(F \backslash \mathbb{A})^5} f(v(y')h, s) dy',$$

$$I_{w_0, \nu_0}^0 = \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{P_3^0(F) \backslash C(\mathbb{A})} \int_{(F \backslash \mathbb{A})^5} \tilde{\varphi}_\pi(g) \tilde{\theta}_\phi^\psi((0|0, y|0)u_0 \varpi_3(g)) f_\tau^{(V, \psi_V^0)}(v(y)w_0 u_0 g, s) dy dg du_0,$$

$$I_{w_0, \nu_0}^1 = \frac{1}{2} \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{SL_2^{\alpha 2}(F) SL_2^{0120}(F)(C \cap U_2)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{(F \backslash \mathbb{A})^5} \tilde{\theta}_\phi^\psi((0|0, y|0)u_0 \varpi_3(g)) f_\tau^{(V, \psi_V^1)}(v(y)w_0 u_0 g, s) dy dg du_0,$$

$$I_{w_0, \nu_0}^a = \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{S_a(F)(C \cap U_2)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) \int_{(F \backslash \mathbb{A})^5} \tilde{\theta}_\phi^\psi((0|0, y|0)u_0 \varpi_3(g)) f_\tau^{(V, \psi_V^a)}(v(y)w_0 u_0 g, s) dy dg du_0,$$

Where  $S_a$  is the stabilizer of  $\psi_V^a$ , described above, and

$$f_\tau^{(V, \psi_V^i)}(h, s) = \int_{(F \backslash \mathbb{A})^5} f_\tau(v(y')h, s) \psi_V^i(v(y)) dy', \quad (i = 0, 1, a).$$



(The integral  $I_{w_0, \nu_0}^1$  can also be expressed as an integral over  $U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A}) \times S_1(F)(C \cap U_2)(F) \backslash C(\mathbb{A})$ , where  $S_1$  is the full stabilizer of  $\psi_V^1$  in  $(C \cap P_2)$ . In the expression above, the factor  $\frac{1}{2}$  is present because the group  $SL_2^{\alpha_2} SL_2^{0120}(C \cap U_2)$  is of index two in this stabilizer.)

Now, it follows from the table in the end of section 4 that  $\mathcal{O}(\tau) = (31^4)$ . For the next step, we shall use this fact to prove that the integrals  $I_{w_0, \nu_0}^{00}$  and  $I_{w_0, \nu_0}^0$  both vanish.

**Proposition 71.** Since  $\tau$  is attached to the orbit  $(31^4)$ ,  $I_{w_0, \nu_0}^{00} = I_{w_0, \nu_0}^0 = 0$ .

*Proof.* We have

$$I_{w_0, \nu_0}^{00} = \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{(C \cap P_2)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) \tilde{\theta}_{\phi|_2}^\psi(u_0 \varpi_3(g)) f_\tau(v(y) w_0 u_0 g, s) dy dg du_0.$$

The parabolic subgroup  $C \cap P_2$  of  $C \cong Sp_6$  maps into the standard parabolic subgroup of  $Sp_{14}$  whose Levi subgroup is isomorphic to  $GL_5 \times Sp_4$ , with the unipotent radical of  $C \cap P_2$  mapping into the unipotent radical of this parabolic subgroup in  $Sp_{14}$ . The function  $\tilde{\theta}_{\phi|_2}^\psi$  is invariant by the unipotent radical of the parabolic subgroup of  $Sp_{14}(\mathbb{A})$  (which lifts into  $\widetilde{Sp}_{14}(\mathbb{A})$ ) and so is  $f_\tau(w_0 u_0 g, s)$ , so this term vanishes by cuspidality of  $\tilde{\varphi}_\pi$ .

Next consider the integral  $I_{w_0, \nu_0}^0$ . Changing variables, and using the formulas for the action of the Weil representation, we have

$$\begin{aligned} & \int_{(F \backslash \mathbb{A})^5} \tilde{\theta}_\phi^\psi((0|0y|0)u_0 \varpi_3(g)) f_\tau^{(V, \psi_V^0)}(v(y) w_0 u_0 g, s) dy \\ &= f_\tau^{(V, \psi_V^0)}(w_0 u_0 g, s) \int_{(F \backslash \mathbb{A})^5} \tilde{\theta}_\phi^\psi((0|0y|0)u_0 \varpi_3(g)) \psi(y_1) dy \\ &= f_\tau^{(V, \psi_V^0)}(w_0 u_0 g, s) \sum_{\xi \in F^2} [\omega_\psi(u_0 \varpi_3(g)) \phi](0, 0, 0, 0, -1, \xi). \end{aligned}$$

Hence, the integral  $I_{w_0, \nu_0}^0$  is equal to

$$\int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{(C \cap P_2)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) \sum_{\xi \in F^2} [\omega_\psi(u_0 \varpi_3(g)) \phi](0, 0, 0, 0, -1, \xi) f_\tau^{(V, \psi_V^0)}(w_0 u_0 g, s) dg du_0.$$

Next, let  $N_1$  denote the unipotent radical of  $(C \cap P_2)$  and factor the integration over this group. Since  $w_0 N_1 w_0^{-1} \subset U_4$ , the function  $f_\tau^{(V, \psi_V^0)}$  is invariant by  $N_1(\mathbb{A})$ .

Now consider the partial theta function  $\tilde{\theta}_1(h) := \sum_{\xi \in F^2} [\omega_\psi(h) \phi](0, 0, 0, 0, -1, \xi)$ , with  $h \in \widetilde{Sp}_{14}(\mathbb{A})$ . We claim that the function  $n_1 \mapsto \tilde{\theta}_1(\varpi_3(n_1)h)$  with  $n_1 \in N_1(\mathbb{A})$ ,  $h \in \widetilde{Sp}_{14}(\mathbb{A})$ , depends only on the inner  $6 \times 6$  block of  $\varpi_3(n_1)$ . To see this, let  $Q_4$  denote the standard parabolic subgroup  $Sp_{14}$  with Levi factor isomorphic to  $GL_4 \times Sp_6$ . Note that  $\varpi_3(n_1)$  is upper triangular, and hence contained in  $Q_4$ . Now, the function  $\tilde{\theta}_1$  is invariant on the left by the unipotent radical of  $Q_4$ , and satisfies  $\tilde{\theta}_1(\text{diag}(g, I_6, {}_t g^{-1})h) = \gamma_{\det g} \tilde{\theta}_1(h)$  for all  $g \in GL_4(\mathbb{A})$

and  $h \in \widetilde{Sp}_{14}(\mathbb{A})$ . This follows from the same argument used to prove lemma 48. The upper left  $4 \times 4$  block of  $\varpi_3(n_1)$  will be unipotent, and hence have determinant one. It follows that the function  $n_1 \mapsto \widetilde{\theta}_1(\varpi_3(n_1)h)$  with  $n_1 \in N_1(\mathbb{A}), h \in \widetilde{Sp}_{14}(\mathbb{A})$ , depends only on the inner  $6 \times 6$  block of  $\varpi_3(n_1)$ , as claimed.

Each element of  $N_1(\mathbb{A})$  can be written uniquely as

$$n_1(r) = x_{0001}(r_{0001})x_{0011}(r_{0011})x_{0111}(r_{0111})x_{0121}(r_{0121})x_{0122}(r_{0122}),$$

and the projection of  $\varpi_3(n_1(r))$  as above onto  $Sp_6$  is

$$\begin{pmatrix} 1 & 0 & 0 & -2r_{0001} & -2r_{0011} & 0 \\ 0 & 1 & 0 & 0 & 0 & -2r_{0011} \\ 0 & 0 & 1 & 0 & 0 & -2r_{0001} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that

$$[\omega_\psi(u_0\varpi_3(n_1(r)g))\phi](0, 0, 0, 0, -1, \xi_6, \xi_7) = \psi(2\xi_7r_{0001} + 2\xi_6r_{0011})[\omega_\psi(u_0\varpi_3(g))\phi](0, 0, 0, 0, -1, \xi_6, \xi_7).$$

First, factor the integration over  $N_2 := U_{0111}U_{0121}U_{0122}$ . Thus  $I_{w_0, \nu_0}^0$  is equal to

$$\int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{(C \cap P_2)(F)N_2(\mathbb{A}) \backslash C(\mathbb{A})} \widetilde{\varphi}_\pi^{N_2}(g) \sum_{\xi \in F^2} [\omega_\psi(u_0\varpi_3(g))\phi](0, 0, 0, 0, -1, \xi) f_\tau^{(V, \psi_V^0)}(w_0u_0g, s) dg du_0,$$

where

$$\widetilde{\varphi}_\pi^{N_2}(g) = \int_{N_2(F) \backslash N_2(\mathbb{A})} \widetilde{\varphi}_\pi(n_2g) dn_2.$$

Now expand  $\widetilde{\varphi}_\pi^{N_2}(g)$  along  $U_{0001}U_{0011}$ . The term corresponding to the trivial character produces a constant term of  $\widetilde{\varphi}_\pi$ , and therefore vanishes by cuspidality. The group  $SL_2^{\alpha_3}$  permutes the nontrivial characters of this group transitively. Choose  $x_{0001}(r_{0001})x_{0011}(r_{0011}) \mapsto \psi(r_{0001})$  as a representative. Then the stabilizer is  $U_{0010}$ . Now factoring the integration over  $U_{0001}U_{0011}(F) \backslash U_{0001}U_{0011}(\mathbb{A})$  picks off a single term in the sum over  $\xi$ , and we have

$$\int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{(C \cap P_2)(F)N_1(\mathbb{A}) \backslash C(\mathbb{A})} \widetilde{\varphi}_\pi^{(N_1, \psi_{N_1})}(g) f_\tau^{(V, \psi_V^0)}(w_0u_0g, s) [\omega_\psi(u_0\varpi_3(g))\phi](0, 0, 0, 0, -1, 0, -1/2) dg du_0,$$

where  $N_1$ , as above, is the unipotent radical of  $C \cap P_2$  and can also be described as  $U_{0001}U_{0011}N_2$ , and

$$\widetilde{\varphi}_\pi^{(N_1, \psi_{N_1})}(g) = \int_{N_1(F) \backslash N_1(\mathbb{A})} \widetilde{\varphi}_\pi(ng) \psi_{N_1}(n) dn \quad \psi_{N_1}(n) = \psi(n_{0001}), n \in N_1(\mathbb{A}).$$

Next, the projection of  $\varpi_3(x_{0010}(r_1)x_{0110}(r_2)x_{0120}(r_3))$  onto  $Sp_6$  is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & r_1 & 2r_2 & 2(r_1r_2 - r_3) & 0 \\ 0 & 0 & 1 & 0 & 2r_2 & 0 \\ 0 & 0 & 0 & 1 & -r_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It follows that

$$[\omega_\psi(u_0\varpi_3(g))\phi](0, 0, 0, 0, -1, 0, -1)$$

is left-invariant by  $U_{0010}U_{0110}U_{0120}$ . To show that  $I_{w_0, \nu_0}^0 = 0$ , we will prove the following.

**Proposition 72.** Since  $\tau$  is attached to  $(31^4)$ , the function

$$f_\tau^{(V, \psi_V^0)}(g, s) := \int_{V(F) \backslash V(\mathbb{A})} f_\tau(vg, s) \psi_V^0(v) dv$$

is left-invariant by  $U_{0100}U_{0110}U_{0120}(\mathbb{A})$ .

Assuming proposition 72, the proof of proposition 71 is as follows: we obtain as inner integration the constant term  $\tilde{\varphi}_\pi^L$  where  $L$  is the unipotent radical of the maximal standard parabolic subgroup of  $Sp_6$  whose Levi part is  $GL_2 \times SL_2$ . By cuspidality this is zero.

*Proof of proposition 72.* Put  $N_3 = U_{0100}U_{0110}U_{0120}$ . Expand  $f_\tau^{(V, \psi_V^0)}(g, s)$  along  $N_3(F) \backslash N_3(\mathbb{A})$ . Under the action of  $SL_3^{\alpha_3}$ , there are infinitely many orbits of characters, represented by

$$n_3(r_1, r_2, r_3) \mapsto 1, \quad n_3(r_1, r_2, r_3) \mapsto \psi(r_3), \quad \text{and} \quad n_3(r_1, r_2, r_3) \mapsto \psi(r_1 + ar_3), \quad a \in F^\times$$

respectively, where  $n_3(r_1, r_2, r_3) = x_{0100}(r_1)x_{0110}(r_2)x_{0120}(r_3)$ . Now, for any  $a \in F^\times$ ,

$$(73) \quad \int_{(F \backslash \mathbb{A})^3} f_\tau^{(V, \psi_V^0)}(n_3(r_1, r_2, r_3)g, s) \psi(r_1 + ar_3) dr$$

is a Fourier coefficient attached to the orbit  $(51^2)$  which is greater than  $(31^4)$ . Hence integral (73) is zero.

In order to prove that

$$(74) \quad \int_{(F \backslash \mathbb{A})^3} f_\tau^{(V, \psi_V^0)}(n_3(r_1, r_2, r_3)g, s) \psi(r_3) dr$$

vanishes, we shall relate it to Fourier coefficients attached to the unipotent orbit  $(3^21)$ , which is also greater than  $(31^4)$ . Let  $N_4$  denote the unipotent subgroup of  $M \cong \text{GSpin}_7$

which projects to

$$\left\{ \text{pr}(n_4) = \begin{pmatrix} 1 & 0 & r_1 & r_2 & r_3 & * & * \\ & 1 & r_4 & r_5 & r_6 & * & * \\ & & 1 & 0 & 0 & -r_6 & -r_3 \\ & & & 1 & 0 & -r_5 & -r_2 \\ & & & & 1 & -r_4 & -r_1 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix} \right\} \subset SO_7.$$

Since this projection is an isomorphism on  $N_4$ , we can use the matrix entries above to define characters of  $N_4$ . An integral

$$\int_{N_4(F) \backslash N_4(\mathbb{A})} f_\tau(n_4 g, s) \psi \left( \sum_{i=1}^6 a_i r_i \right) dn_4$$

is a Fourier coefficient attached to  $(3^2 1)$  if and only if the character  $n_4 \mapsto \psi \left( \sum_{i=1}^6 a_i r_i \right)$  is in general position, which is the case if and only if the matrix  $\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}$  has rank 2. Now let  $N_4^0$  denote the subgroup of  $N_4$  defined by setting  $r_4 = r_5 = 0$ . Use  $r_1, r_2, r_3, r_6$  as coordinates on  $N_4^0$  in the obvious way. We claim that

$$(75) \quad \int_{N_4^0(F) \backslash N_4^0(\mathbb{A})} f_\tau(n_4^0 g, s) \psi(a_1 r_1 + a_6 r_6) dn_4^0$$

vanishes identically whenever  $a_1 a_6 \neq 0$ . Indeed, this integral is left-invariant by the  $F$ -points of two dimensional subgroup  $U_{0010} U_{0110}$  corresponding to the omitted variables  $r_4$  and  $r_5$ . One may perform a Fourier expansion along this group, and every term will produce a Fourier coefficient of  $f_\tau$  attached to  $(3^2 1)$ . Thus integral (75) is zero for all choice of data, whenever  $a_1 a_6 \neq 0$ .

In (75), use the left invariance property of  $f_\tau$  to obtain  $f_\tau(n_4^0 g, s) = f_\tau(w[2]n_4^0 g, s)$ . Conjugating  $w[2]$  to the right, we obtain an integral over  $w[2]N_4^0(F) \backslash N_4^0(\mathbb{A})w[2]^{-1}$ . This integral is clearly zero for all choice of data, and, for suitable nonzero  $a_1, a_6$ , is also an inner integration to integral (74). Thus we conclude that integral (74) is zero for all choice of data.  $\square$

$\square$

We now treat  $I_{w_0, \nu_0}^1$ . Let

$$\psi_{U^{w_0}}^1((0|0, 0, y|0)) = \psi(-y_3), \quad y = (y_1, y_2, y_3, y_4, y_5) \in \mathbb{A}^5$$

be the unique character of  $U^{w_0}(\mathbb{A})$  such that  $\psi_{U^{w_0}}^1(u) \psi_V^1(v(w_0 u w_0^{-1}))$  is trivial.

Using the action of the Weil representation, we obtain that

$$\int_{(F \backslash \mathbb{A})^5} \tilde{\theta}_\phi^\psi((0|0, y|, 0)u\tilde{h}) \psi_{U^{w_0}}^1(y) dy = \sum_{\xi \in F^2} \omega_\psi(u\tilde{h}) \phi(0, 0, -1, 0, 0, \xi),$$

for all  $\tilde{h} \in \widetilde{Sp}_{14}(\mathbb{A})$ ,  $u \in \mathcal{H}_{14}(\mathbb{A})$ . Denote the left hand side by  $\tilde{\theta}_\phi^{(U^{w_0}, \psi_{U^{w_0}}^1)}(u\tilde{h})$ .

Changing variables in  $f_\tau^{V, \psi_V^1}$  then proves that

$$I_{w_0, \nu_0}^1 = \frac{1}{2} \int_{U_{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{SL_2^{\alpha_2}(F) SL_2^{0120}(F)(C \cap U_2)(F) \backslash C(\mathbb{A})} \tilde{\varphi}_\pi(g) \tilde{\theta}_\phi^{(U^{w_0}, \psi_{U^{w_0}}^1)}(u_0 \varpi_3(g)) f_\tau^{(V, \psi_V^1)}(w_0 u_0 g, s) dg du_0.$$

Next we factor the integration over  $N_1 := U_{0001}U_{0011}U_{0111}U_{0121}U_{0122}$ . This group is the unipotent radical of a standard parabolic subgroup of  $C \cong Sp_6$ , having Levi subgroup isomorphic to  $GL_1 \times Sp_4$ . Further, it is isomorphic to the Heisenberg group  $\mathcal{H}_5$ .

Set

$$n'_1(r) = x_{0111}(r_{0111})x_{0121}(r_{0121})x_{0122}(r_{0122}), \quad n''_1(r) = x_{0001}(r_{0001})x_{0011}(r_{0011}).$$

Then one may compute  $\varpi_3(n'_1(r))$  and  $\varpi_3(n''_1(r))$ . Each will be of the form  $\begin{pmatrix} u & b & c \\ & h & b' \\ & & {}_t u^{-1} \end{pmatrix}$ , with  $u \in SL_2, h \in Sp_{10}, b \in \text{Mat}_{2 \times 10}$ , etc., and it follows from lemma 48, that we only need the middle  $10 \times 10$  block,  $h$ . The projection of  $\varpi_3(n'_1(r))$  onto  $Sp_{10}$  is

$$\begin{pmatrix} I_5 & X'(r) \\ & I_5 \end{pmatrix}, \quad X'(r) = \begin{pmatrix} 2r_{0111} & 2r_{0121} & 0 & 0 & -2r_{0122} \\ 0 & -2r_{0111} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2r_{0111} & 2r_{0121} \\ 0 & 0 & 0 & 0 & 2r_{0111} \end{pmatrix},$$

whence, by the action of the Weil representation,

$$[\omega_\psi(\varpi_3(n'_1(r)))\phi](0, 0, -1, 0, 0, \xi_1, \xi_2) = \psi(-r_{0122} - 2\xi_1 r_{0121} - 2\xi_2 r_{0111})\phi(0, 0, -1, 0, 0, \xi_1, \xi_2).$$

The projection of  $\varpi_3(n''_1(r))$  onto  $Sp_{10}$  is

$$\begin{pmatrix} I_5 & X''_1(r) \\ & I_5 \end{pmatrix} \begin{pmatrix} h''_1(r) \\ {}_t h''_1(r)^{-1} \end{pmatrix}, \quad h''_1(r) := \begin{pmatrix} 1 & 0 & 0 & r_{0001} & r_{0011} \\ 0 & 1 & 0 & 0 & r_{0001} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X''_1(r) := \begin{pmatrix} 0 & 0 & -2r_{0001}r_{0011} & 0 & 0 \\ 0 & 0 & -r_{0001}^2 & 0 & 0 \\ -2r_{0001} & -2r_{0011} & 0 & -r_{0001}^2 & -2r_{0001}r_{0011} \\ 0 & 0 & -2r_{0011} & 0 & 0 \\ 0 & 0 & -2r_{0001} & 0 & 0 \end{pmatrix},$$

whence

$$[\omega_\psi(\varpi_3(n''_1(r)))\phi](0, 0, -1, 0, 0, \xi_1, \xi_2) = \phi(0, 0, -1, 0, 0, \xi_1 - r_{0001}, \xi_2 - r_{0011}).$$

Define  $\phi_0 \in S(\mathbb{A}^2)$  by  $\phi_0(x_1, x_2) = \phi(0, 0, -1, 0, 0, x_1, x_2)$ . Then the above shows that

$$(76) \quad [\omega_\psi \circ \varpi_3(n_1)]\phi(0, 0, -1, 0, 0, x_1, x_2) = [\omega_{\psi_{-2}}^{(4)} \circ \iota(n_1)]\phi_0(x_1, x_2).$$

for a suitable isomorphism  $\iota : N_1 \rightarrow \mathcal{H}_5$ . Here  $\omega_{\psi_{-2}}^{(4)}$  denotes the Weil representation of  $\mathcal{H}_5(\mathbb{A}) \rtimes \widetilde{Sp}_4(\mathbb{A})$  defined using the additive character  $\psi_{-2}(x) = \psi(-2x)$  in place of  $\psi$ . By uniqueness of the extension of  $\omega_{\psi_{-2}}^{(4)}$  from  $\mathcal{H}_5$  to  $\mathcal{H}_5(\mathbb{A}) \rtimes \widetilde{Sp}_4(\mathbb{A})$ , it follows that (76) holds for  $g \in \widetilde{Sp}_4(\mathbb{A})$  (i.e., the preimage in  $\widetilde{C}(\mathbb{A})$  of  $Sp_4^{\{\alpha_2, \alpha_3\}}(\mathbb{A})$ ) as well.

Now, the identity component of the stabilizer of  $\psi_{U^{w_0}}^1$  in  $Sp_4^{\{\alpha_2, \alpha_3\}}$  is  $SL_2^{\alpha_2} SL_2^{0120}$ . In order to complete our analysis of this case, we show that  $f_\tau^{(U^{w_0}, \psi_{U^{w_0}}^1)}$  is left-invariant by the  $\mathbb{A}$  points of this group. To prove that, and also the analogous statement related to the integrals  $I_{w_0, \nu_0}^a$ , we prove the following.

**Proposition 77.** Since  $\tau$  is attached to  $(31^4)$ , the function

$$f_\tau^{(V, \psi_V^a)}(g, s) = \int_{V(F) \backslash V(\mathbb{A})} f_\tau(vg, s) \psi_V(v) dv$$

satisfies

$$f_\tau^{(V, \psi_V^a)}(hg, s) = f_\tau^{(V, \psi_V^a)}(g, s) \quad (\forall g \in C(\mathbb{A}), \text{ and } h \in S_a(\mathbb{A})),$$

for all  $a \neq 0$ , where  $S_a$  is the stabilizer of  $\psi_V^a$  in  $Sp_4^{\{\alpha_2, \alpha_3\}}$ .

*Proof.* The group  $S_a(\mathbb{A})$  is generated by  $S_a(F)$  and the two dimensional maximal unipotent subgroup  $L(\mathbb{A}) := S_a(\mathbb{A}) \cap U_{0100} U_{0110} U_{0120}(\mathbb{A})$ , so it suffices to prove that  $f_\tau^{(V, \psi_V^a)}$  is invariant by this unipotent group. Clearly, we may expand  $f_\tau^{(V, \psi_V^a)}$  along  $L(F) \backslash L(\mathbb{A})$ . We claim that every term other than the constant term vanishes.

Recall that, by hypothesis,  $\tau$  is attached to the unipotent orbit  $(31^4)$ , and thus does not support any Fourier coefficient which is attached to the orbit  $(3^2 1)$ . The set of such Fourier coefficients was described before equation (75).

Let  $V_1$  denote the product of all groups  $U_\alpha \subset V$  except for  $U_{1000}$ . Let  $N_4$  denote the unipotent radical of the maximal parabolic subgroup of  $\mathrm{GSpin}_7$  with Levi isomorphic to  $GL_2 \times \mathrm{GSpin}_3$ , and  $R_1$  denote the product of  $V_1$  and  $L$ . Then  $R_1$  is a codimension 1 subgroup of  $N_4$ . To be precise: there is a column vector  $c_a = (1, 0, -a)$  or  $(0, 1, 0)$ , such that

$$\psi_V^a \left( \begin{pmatrix} 1 & x & * \\ & I_5 & -_t x \\ & & 1 \end{pmatrix} \right) = \psi \left( (x_2 \ x_3 \ x_4) \cdot c_a \right),$$

and

$$R_1 = \left\{ \left( \begin{pmatrix} I_2 & X & * \\ & I_3 & -_t X \\ & & I_2 \end{pmatrix} : X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}, (x_4 \ x_5 \ x_6) \cdot c_a = 0 \right) \right\}.$$

Here we identify unipotent elements of  $M_4$  with their images in  $SO_7$ . This follows from the fact that  $I_7 + re'_{12} \in V$  and  $L$  preserves  $\psi_V^a$ . Now, let  $\psi_{R_1}$  be any character such that

$\psi_{R_1}|_{V_1(\mathbb{A})} = \psi_V|_{V_1(\mathbb{A})}$ , and  $\psi_{R_1}|_{L(\mathbb{A})}$  is nontrivial. We claim that

$$f_\tau^{(R_1, \psi_{R_1})} := \int_{R_1(F) \backslash R_1(\mathbb{A})} f_\tau(r_1 g, s) \psi_{R_1}(r_1) dr_1.$$

vanishes identically. Indeed, one may expand  $f_\tau^{(R_1, \psi_{R_1})}$  on  $R_1(\mathbb{A})N_4(F) \backslash N_4(\mathbb{A})$  and obtain an expression as a sum of terms, each of which is an integral on  $N_4(F) \backslash N_4(\mathbb{A})$  against a character whose restriction to  $R_1$  is  $\psi_{R_1}$ . But it is clear from the remarks above that every character of  $N_4$  which restricts to  $\psi_{R_1}$  is in general position. As explained before equation (75), such an integral is a Fourier coefficient attached to  $(3^2 1)$ , and therefore vanishes identically.  $\square$

It follows that with the notation as above

$$I_{w_0, \nu_0}^1 = \frac{1}{2} \int_{U^{w_0}(\mathbb{A}) \backslash U(\mathbb{A})} \int_{N_1(F)SL_2^{\alpha_2}(F)SL_2^{0120}(F) \backslash C(\mathbb{A})} \widetilde{\varphi}_\pi(g) f_\tau^{(V, \psi_V)}(u_0 g, s) \\ \sum_{\xi \in F^2} [\omega_\psi(u_0 \varpi_3(g)) \phi](0, 0, -1, 0, 0, \xi_6, \xi_7) dg du_0.$$

The inner period in this case is

$$\int_{\mathcal{H}_5(F) \backslash \mathcal{H}_5(\mathbb{A})} \int_{SL_2(F) \backslash SL_2(\mathbb{A})} \int_{SL_2(F) \backslash SL_2(\mathbb{A})} \widetilde{\varphi}_\pi \left( u \begin{pmatrix} 1 & & \\ & \alpha(g_1, g_2) & \\ & & 1 \end{pmatrix} \right) \theta_\phi^{\psi_2}(u \alpha(g_1, g_2)) dg_1 dg_2 du,$$

where  $\widetilde{\varphi}_\pi$  is a genuine cusp form  $\widetilde{Sp}_6(\mathbb{A}) \rightarrow \mathbb{C}$ ,  $\theta_\phi^{\psi_2}$  is a theta function on  $\mathcal{H}_5(\mathbb{A}) \rtimes \widetilde{Sp}_4(\mathbb{A})$  defined using the character  $\psi_2(x) = \psi(2x)$  and we have identified  $\mathcal{H}_5$  with

$$\left\{ \begin{pmatrix} 1 & X & y \\ & I_4 & X' \\ & & 1 \end{pmatrix} \right\} \subset Sp_6.$$

Also  $\alpha : SL_2 \times SL_2 \rightarrow Sp_4$  is given by

$$\alpha \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g_2 \right) = \begin{pmatrix} a & & b \\ & g_2 & \\ c & & d \end{pmatrix}.$$

The analysis of  $I_{w_0, \nu_0}^a$  nonsquare is similar. Let

$$\psi_{U^{w_0}}^a((0|0, 0, y|0)) = \psi(-y_2 + ay_4)$$

be the unique character of  $U^{w_0}(\mathbb{A})$  such that  $\psi_{U^{w_0}}^1(u) \psi_V^1(v(w_0 u w_0^{-1}))$  is trivial. Then the integral over  $V(F) \backslash V(\mathbb{A})$  picks off a partial theta series

$$\theta_\phi^{(U^{w_0}, \psi_{U^{w_0}}^1)}(h) := \sum_{\xi \in F^2} \omega_\psi(h) \phi(0, -1, 0, a, 0, \xi).$$

With notation as before, similar computations show that

$$[\omega_\psi(\varpi_3(n'_1(r))) \phi](0, -1, 0, a, 0, \xi_1, \xi_2) = \psi(-ar_{0122} + 2a\xi_1 r_{0111} - 2\xi_2 r_{01s1}) \phi(0, -1, 0, a, 0, \xi_1, \xi_2),$$

whence

$$[\omega_\psi(\varpi_3(n_1''(r)))\phi](0, -1, 0, a, 0, \xi_1, \xi_2) = \phi(0, -1, 0, a, 0, \xi_1 + r_{0011}, \xi_2 - ar_{0011}).$$

This is still essentially the action of the Weil representation of  $\mathcal{H}(\mathbb{A}) \times \widetilde{Sp}_4(\mathbb{A})$ . Determined by the character  $\psi_{-2a}(x) = \psi(-2ax)$ .

We conclude that this case does not appear to be of open orbit type.

## 10. APPENDIX

The purpose of this appendix is to elaborate slightly on the earlier remark that the Fourier coefficient  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  of an automorphic form  $\varphi$  will not, in general, be an automorphic form, because it will not, in general, be  $\mathfrak{z}$ -finite. This should be something that one can prove using purely local methods, but here it is more convenient to point out that the existence of an integral representation which involves some Fourier coefficient tends to suggest that the Fourier coefficient is far from  $\mathfrak{z}$ -finite. Indeed, it is, in a sense, spread out all over the spectrum of the operators in  $\mathfrak{z}$ .

To illustrate this, consider the integral representation for the exterior square  $L$  function of a cuspidal automorphic representation of  $GL_{2n}(\mathbb{A})$ , which was given in [J-S1]. This construction consists of integrating  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  against an Eisenstein series  $E(g, s)$  defined on the group  $GL_n$ , with  $\varphi$  itself being a cuspform in the space of an irreducible cuspidal automorphic representation of the group  $GL_{2n}$ . The function  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  inherits the rapid decay of the cusp form  $\varphi$ . Hence the integral

$$\int_{Z(\mathbb{A})GL_n(F) \backslash GL_n(\mathbb{A})} \varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}(g) E(g, s) dg$$

is absolutely convergent for all  $s$  where  $E(g, s)$  has no poles, uniformly for  $s$  in a compact set. Reversing order and making a change of variable,

$$\begin{aligned} & \int_{Z(\mathbb{A})GL_n(F) \backslash GL_n(\mathbb{A})} \lim_{t \rightarrow 0} \varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}(g \exp(tX)) E(g, s) dg \\ &= \int_{Z(\mathbb{A})GL_n(F) \backslash GL_n(\mathbb{A})} \varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}(g) \lim_{t \rightarrow 0} E(g \exp(tX), s) dg \end{aligned}$$

for all  $X \in \mathfrak{gl}(2, \mathbb{R})$ . Consequently, even though  $E(g, s)$  is not  $L^2$ , one may still think of this integral as a pairing  $\langle \cdot, \cdot \rangle$  with respect to which the elements of the center of the universal enveloping algebra are “self-adjoint.” Now for concreteness, assume  $n = 2$  and  $F = \mathbb{Q}$ , and take  $\Delta$  the Laplace-Beltrami operator. Then  $\Delta E(g, s) = s(1-s)$  for all  $s \in \mathbb{C}$ . Suppose that  $\varphi^{(U_{\mathcal{O}}, \psi_{U_{\mathcal{O}}})}$  is  $\mathfrak{z}$ -finite. Then it is killed by a polynomial  $P(\Delta)$  in  $\Delta$ . But then

$$0 = \langle P(\Delta)\varphi, E(\cdot, s) \rangle = \langle \varphi, P(\Delta)E(\cdot, s) \rangle = P(s(1-s))\langle \varphi, E(\cdot, s) \rangle$$



for all values of  $s$ . Since  $P(s(1-s))$  has only finitely many zeros, and  $\langle \varphi, E(\cdot, s) \rangle$  is nonzero for  $\Re(s)$  large, we have a contradiction.

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